Rule Calculus: Semantics, Axioms and Applications

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Abstract. We consider the problem of how a default rule can be deduced from a default theory. For this purpose, we propose an axiom system which precisely captures the deductive reasoning about default rules. We show that our axiomatic system is sound and complete under the semantics of the logic of here-and-there. We also study other important properties such as substitution and monotonicity of our system and prove the essential decision problem complexity. Finally, we discuss applications of our default rule calculus to various problems.

1 Introduction

Default logic is one of the predominant approaches for nonmonotonic reasoning. Many research topics related to default logic have been considerably studied including extensions, variations and alternatives [1–3] of Reiter's original definition [4], computational issues [5, 6] and so on.

However, one problem in default logic has been neglected in previous research. That is, how can we deduce a default rule from a default theory? In other words, in which sense can we say that a default rule is a consequence of a given default theory? This problem of rule deduction is of special interests from both theoretical and practical viewpoints. For instance, we may consider whether we can have a deductive system to formalize reasoning about default rules, and also implement a nonmonotonic knowledge system for more complex decision making where a decision could be a default rule. Quite obviously, to achieve such goals, the first fundamental task is that we should develop a logic or calculus for default rule reasoning.

In this paper, we propose a logical calculus, called *default rule calculus* (rule calculus for short), to address the problem of rule deduction. We first extend the logic of here-and-there to define a model-theoretical semantics for rule calculus, and discuss its relationships to the extension semantics. Then we define an axiom system, which extends both classical propositional calculus and the intermediate logic G3 (Gödel's 3-valued logic) [7], and prove its soundness and completeness. We further investigate some important properties of our system such as substitution and monotonicity, and prove the essential decision problem complexity. Finally, we discuss how our work can be applied to various problems such as the extension of generality among default rules and revision of nonmonotonic knowledge bases.

The reasons why we use the logic of here-and-there are as follows. Firstly, it is well studied in philosophical logic and it also has a simple axiomatic counterpart, namely G3

[7]. Secondly, it is proven to be a very useful foundation of answer set programming [2, 8–10]. As pointed out in [3], answer set programming is a special case of default logic by restricting the propositional formulas to atoms. Thus, the extended version of the logic of here-and-there should, analogously, serve as a foundation of default logic. Thirdly, the logic of here-and-there naturally captures [2, 11] the notion of strong equivalence [11], which is argued to be the notion of ”real equivalence” among answer set programs. As a consequence, it will also capture real equivalence among default rules.

In this paper, we will use general default logic [3] as a basis for the development of default rule calculus. Reasons for this are of three aspects: firstly, general default logic is a generalization of Reiter's default logic [4], Gelfond et al.'s disjunctive default logic [1] and Turner's nested default logic [2], which provides the most generalized default reasoning in the default logics paradigm; secondly, the syntax of general default logic is defined as arbitrary compositions of propositional formulas and rule connectives; finally, its extension semantics is defined in a very simple way as that for answer set programming [9].

The rest of the paper is organized as follows. In Section 2, we briefly review the syntax and semantics of general default logic, and then define the semantics of rule calculus. In Section 3, we present an axiom system for rule calculus and prove its soundness and completeness result. We then study relevant important properties of our axiom system for default rule calculus in Section 4. In Section 5, we discuss possible applications of rule calculus. Finally, in Section 6 we conclude the paper with some remarks.

2 Rule calculus: syntax and semantics

To begin with, we recall some basic notions of classical propositional logic. The classical propositional language $\mathcal{L}$ is defined recursively by a set $Atom$ of atoms (or primitive propositions, variables) and a set of classical connectives $\bot$, $\rightarrow$ and $\neg$. Other connectives, such as $\top$, $\wedge$, $\vee$, $\leftrightarrow$, are defined as usual. Literals are atoms and their negations. The satisfaction relation $\models$ is defined as usual. A set of formulas in $\mathcal{L}$ is said to be a *theory* iff it is closed under classical entailment. Moreover, it is inconsistent iff it contains both a formula F and $\neg F$, otherwise, it is consistent. Let Γ be a set of formulas, by $Th(\Gamma)$ we denote the theory containing all formulas entailed by Γ. For convenience, we also use a set of formulas Γ to denote a theory T if $T = Th(\Gamma)$.

The language $\mathcal{R}$ of general default logic [3] is defined upon $\mathcal{L}$ by adding a set of rule connectives $\Rightarrow$, $\,\&\,$ and $\mid$ recursively:

$$R ::= F \mid R \Rightarrow R \mid R \,\&\, R \mid R \,|\, R,$$

where $F \in \mathcal{L}$. $-R$ and $R_1 \Leftrightarrow R_2$ are considered as shorthand of $R \Rightarrow \bot$ and $(R_1 \Rightarrow R_2) \,\&\, (R_2 \Rightarrow R_1)$ respectively. The order of priority for these connectives are

$$\{\neg\} > \{\wedge, \vee\} > \{\rightarrow, \leftrightarrow\} > \{-\} > \{\,\&\,, \,|\,\} > \{\Rightarrow, \Leftrightarrow\}.$$

Formulas in $\mathcal{R}$ are called *rules*, whilst formulas in $\mathcal{L}$ are called *facts*. A *rule base* is a set of rules. The satisfaction relation $\models$ between a theory T and a rule R is defined recursively as follows:

– If R is a fact, then $T \models R$ iff $R \in T$.
– $T \models R \,\&\, S$ iff $T \models R$ and $T \models S$;
– $T \models R \mid S$ iff $T \models R$ or $T \models S$;
– $T \models R \Rightarrow S$ iff $T \not\models R$ or $T \models S$.

Hence, if T is consistent, then $T \models -R$ iff $T \not\models R$. If T is inconsistent, then for every rule R, $T \models R$. We say that T is a *model* of R iff $T \models R$.

The extension semantics of general default logic defined in [3] is not defined as the same as Reiter's original definition [4]. However, it is defined in a reduction-style similarly to that of answer set programming [9]. The *reduct* of a rule R relative to a theory T, denoted by R^T, is the rule obtained from R by replacing every maximal subrule[1] of R which is not satisfied by T with $\bot$. The reduct of a rule base relative to a theory is defined as the set of reducts of its rules relative to this theory. A theory T is said to be an *extension* of a rule base Δ iff it is the minimal (in the sense of set inclusion) theory satisfying Δ^T.

As shown in [3], Reiter's default logic [4] in propositional case is a special case of general default logic by restricting the rules to the following form

$$F \,\&\, -G_1 \,\&\, \ldots \,\&\, -G_n \Rightarrow H,$$

where $n \geq 0$, F, $G_i, (1 \leq i \leq n)$ and H are facts. Yet, under the context of Reiter's default logic, this form is represented as

$$\frac{F : M(\neg G_1), \ldots, M(\neg G_n)}{H}.$$

Similarly, both Gelfond et al.'s disjunctive default logic [1] and Turner's nested default logic [2] are also special cases of general default logic.

Here, we adopt Turner's (Section 7 in [2]) extended notion of Heyting's logic of here-and-there, introduced by Pearce [10] into answer set programming, as the basic semantics for general default logic. An *HT-interpretation* is a pair $\langle T_1, T_2 \rangle$, where T_1 and T_2 are theories such that $T_1 \subseteq T_2$. The satisfaction relation $\models$[2] between an HT-interpretation $\langle T_1, T_2 \rangle$ and a rule R is defined recursively:

– for a fact F, $\langle T_1, T_2 \rangle \models F$ iff $F \in T_1$;
– $\langle T_1, T_2 \rangle \models R_1 \,\&\, R_2$ iff $\langle T_1, T_2 \rangle \models R_1$ and $\langle T_1, T_2 \rangle \models R_2$;
– $\langle T_1, T_2 \rangle \models R_1 \mid R_2$ iff $\langle T_1, T_2 \rangle \models R_1$ or $\langle T_1, T_2 \rangle \models R_2$;
– $\langle T_1, T_2 \rangle \models R_1 \Rightarrow R_2$ iff
  1. $\langle T_1, T_2 \rangle \not\models R_1$ or $\langle T_1, T_2 \rangle \models R_2$, and
  2. $T_2 \models R_1 \Rightarrow R_2$.

We say that $\langle T_1, T_2 \rangle$ is an *HT-model* of a rule R iff $\langle T_1, T_2 \rangle \models R$. We say that a rule base Δ *implies* a rule R, denoted by $\Delta \models R$, iff all HT-models of Δ are also HT-models of R. It is easy to see that the HT-interpretation $\langle \bot, \bot \rangle$ is a model of all rules. We say

[1] The subrule relation is defined recursively: a) R_1 is a subrule of R_1, and b) R_1 and R_2 are subrules of $R_1 \,\&\, R_2$, $R_1 \mid R_2$ and $R_1 \Rightarrow R_2$.
[2] For convenience, we overload the notation $\models$ in this paper.

that a rule R is a *rule contradiction* iff $\langle\bot,\bot\rangle$ is the only HT-model of R. We say that a rule R is a *rule tautology* iff every HT-interpretation is an HT-model of R.

Intuitively, a theory T is a possible set of information of an agent about the world, and a rule R is a description or constraint about the information of the world. $T \models R$ means that the set T of information obeys (or does not violate) the constraint R; T is an extension of R means that T is one of the possible sets of information can be derived by giving the only constraint R. Given a rule base Δ and a rule R, $\Delta \models R$ means that the set of constraints Δ is more powerful than the constraint R. In other words, R can be eliminated by giving Δ.

Example 1. Consider the notorious bird-fly example. The statement "birds normally fly" can be represented as a default rule $bird\&-\neg fly \Rightarrow fly$. Given an instance of bird, represented as a fact $bird$, by the extension semantics, the only extension of the rule base $\{bird \,\&\, -\neg fly \Rightarrow fly, bird\}$ is $\{bird, fly\}$. However, fly is not a rule consequence of the rule base $\{bird \,\&\, -\neg fly \Rightarrow fly, bird\}$ since $\langle\{bird\}, \{bird, \neg fly\}\rangle$ is an HT-model of $\{bird \,\&\, -\neg fly \Rightarrow fly, bird\}$ but not an HT model of $\{bird, fly\}$. This shows a difference between the extension semantics and the HT-semantics.

Similarly, $bird \,\&\, -\neg fly \Rightarrow fly \not\models bird \Rightarrow fly$. However, one can check that all HT-models of $bird \Rightarrow fly$ are also HT-models of $bird \,\&\, -\neg fly \Rightarrow fly$. Therefore, $bird \Rightarrow fly \models bird\& -\neg fly \Rightarrow fly$. This means that, intuitively, the statement "birds normal fly" is strictly weaker than the statement "birds fly".

Example 2. Let p_1, p_2 and p_3 be three atoms. Consider the rule base $\{p_1 \,\&\, -\neg p_2 \Rightarrow p_2, p_2 \,\&\, -\neg p_3 \Rightarrow p_3\}$, which has an HT-model $\langle\{p_1\}, \{p_1, \neg p_2\}\rangle$. However, this HT-interpretation is not an HT-model of $p_1 \,\&\, -\neg p_3 \Rightarrow p_3$. This shows that $\{p_1 \,\&\, -\neg p_2 \Rightarrow p_2, p_2 \,\&\, -\neg p_3 \Rightarrow p_3\} \not\models p_1 \,\&\, -\neg p_3 \Rightarrow p_3$.

The extension semantics and HT-semantics of general default logic are closely related.

Proposition 1. *Let T_1 and T_2 be two consistent theories such that $T_1 \subseteq T_2$ and R a rule.*

- $T_1 \models R$ *iff* $\langle T_1, T_1\rangle \models R$.
- *If* $\langle T_1, T_2\rangle \models R$, *then* $T_2 \models R$.
- $\langle T_1, T_2\rangle \models -R$ *iff* $T_2 \models -R$.

Proposition 2. *Let Δ be a rule base and F a fact. If $\Delta \models F$, then F is in all extensions of Δ.*

However, the converse of Proposition 2 does not hold in general. For instance, $\{p_1\}$ is the unique extension of $-p_2 \Rightarrow p_1$. Thus, p_1 is in all extensions of $-p_2 \Rightarrow p_1$. However, $-p_2 \Rightarrow p_1 \not\models p_1$ since $\langle\{p_2\}, \{p_2\}\rangle$ is an HT-model of $-p_2 \Rightarrow p_1$ but not an HT-model of p_1.

Proposition 3. *Let T_1 and T_2 be two theories such that $T_1 \subseteq T_2$ and Δ a rule base. $\langle T_1, T_2\rangle$ is an HT-model of Δ iff $T_1 \models \Delta^{T_2}$.*

Proposition 4. *Let T be a theory and Δ a rule base. T is an extension of Δ iff $\langle T, T \rangle$ is an HT-model of Δ, and for all theories $T_1 \subset T$, $\langle T_1, T \rangle$ is not an HT-model of Δ.*

The notion of strong equivalence, introduced by [11] into answer set programming, plays an important role from both a theoretical and a practical viewpoints. A similar notion is introduced into default logic in [2]. We say that two rules R_1 and R_2 are *strongly equivalent*, denoted by $R_1 \equiv R_2$, iff for all other rules R_3, $R_1 \,\&\, R_3$ has the same set of extensions as $R_2 \,\&\, R_3$. Strong equivalence can also be defined in another way. That is, two rules R_1 and R_2 are strongly equivalent iff for all other rules R_3, R_3 has the same set of extensions as $R_3(R_1/R_2)$, where $R_3(R_1/R_2)$ is the rule obtained from R_3 by replacing every occurrence of R_1 in R_3 with R_2 simultaneously. It is clear that the notion of strong equivalence can be extended for the cases of rule bases.

In fact, strong equivalence in general default logic can be captured in the logic of here-and-there.

Proposition 5. *Let R_1 and R_2 be two rules. R_1 and R_2 are strongly equivalent iff they have the same set of HT-models in the logic of here-and-there. That is, $R_1 \equiv R_2$ iff $\models R_1 \Leftrightarrow R_2$.*

Since general default logic is both an extension of general logic programming and nested default logic, Proposition 5 is a generalization of both Proposition 2 in [9] and Theorem 3 in [2]. As a consequence of Proposition 5, checking whether a rule is implied by a rule base can be reduced to checking whether two rule bases are strongly equivalent.

Corollary 1. *Let Δ be a rule base and R a rule. $\Delta \models R$ iff $\Delta \cup \{R\}$ is strongly equivalent to Δ.*

Corollary 1 indicates the intuition behind rule deduction, that is, a rule R is a consequence of a rule base Δ means that R provides no more information by giving Δ. In other words, R can be eliminated by giving Δ.

3 Rule calculus: axiom system

In this section, we propose an axiom system for default rule calculus and prove the soundness and completeness results.

Axioms The axioms of rule calculus are:

A1 all tautologies in classical propositional logic.
A2 $(F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)$, where F_1 and F_2 are two facts.
A3 $R_1 \Rightarrow (R_2 \Rightarrow R_1)$.
A4 $(R_1 \Rightarrow (R_2 \Rightarrow R_3)) \Rightarrow ((R_1 \Rightarrow R_2) \Rightarrow (R_1 \Rightarrow R_3))$.
A5 $R_1 \Rightarrow (R_2 \Rightarrow (R_1 \,\&\, R_2))$.
A6 $R_1 \,\&\, R_2 \Rightarrow R_1$; $R_1 \,\&\, R_2 \Rightarrow R_2$.
A7 $R_1 \Rightarrow R_1 \mid R_2$; $R_2 \Rightarrow R_1 \mid R_2$.
A8 $(R_1 \Rightarrow R_3) \Rightarrow ((R_2 \Rightarrow R_3) \Rightarrow (R_1 \mid R_2 \Rightarrow R_3))$.
A9 $(R_1 \Rightarrow R_2) \Rightarrow ((R_1 \Rightarrow -R_2) \Rightarrow -R_1)$.
A10 $R_1 \mid (R_1 \Rightarrow R_2) \mid -R_2$.

Rules The only[3] inference rule of rule calculus is

Rule Modus Ponens from R_1 and $R_1 \Rightarrow R_2$ to infer R_2.

Axiom 1 simply means that all classical tautologies are also rule tautologies; Axiom 2 is for bridging the gap between facts and rules; Axiom 3-9, together with Rule Modus Ponens, are generalization of the axiom system of intuitionistic logic [12]; Axiom A10 is the extended version of an additional axiom in the intermediate logic G3. That is, Axiom 3-10 and Rule Modus Ponens are generalization of the axiom system of G3 [10].

A rule R is said to be a *consequence* of a rule base Δ, denoted by $\Delta \vdash R$, iff there is a sequence of rules $R_1, \ldots, R_n$ such that $R_n = R$ and for each $i, (1 \leq i \leq n)$, either a) R_i is an instance of axiom, or b) R_i is in Δ, or c) R_i is obtained by an inference rule from some proceeding rules in this sequence. Such a sequence is called a *proof* (or *deduction*) of R from Δ. rules in Δ are called *premises*. A rule R is said to be a *rule theorem*, denoted by $\vdash R$, iff there exists a proof of R from the empty rule base. We use $\Delta \nvdash R$ to denote it is not the case that $\Delta \vdash R$.

As an example, we prove the following rule theorem.

Proposition 6. $\vdash (F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2)$, *where* F_1 *and* F_2 *are two facts.*

Proof. We construct a proof as follows:

1. $(\neg F_2 \rightarrow \bot) \Rightarrow (\neg F_2 \Rightarrow \bot)$ by A2,
2. $(F_2 \Rightarrow -\neg F_2) \Rightarrow (F_1 \Rightarrow (F_2 \Rightarrow -\neg F_2))$ by A3,
3. $F_1 \Rightarrow (F_2 \Rightarrow -\neg F_2)$ by 1, 2 and RMP,
4. $(F_1 \Rightarrow (F_2 \Rightarrow -\neg F_2)) \Rightarrow ((F_1 \Rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2))$ by A4,
5. $(F_1 \Rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2)$ by 3, 4 and RMP,
6. $((F_1 \Rightarrow F_2) \Rightarrow ((F_1 \Rightarrow -\neg F_2)) \Rightarrow ((F_1 \rightarrow F_2) \Rightarrow ((F_1 \Rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2))))$ by A3,
7. $(F_1 \rightarrow F_2) \Rightarrow ((F_1 \Rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2))$ by 5, 6 and RMP,
8. $((F_1 \rightarrow F_2) \Rightarrow ((F_1 \Rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2))) \Rightarrow (((F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)) \Rightarrow ((F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2)))$ by A4,
9. $((F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)) \Rightarrow ((F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2))$ by 7, 8 and RMP,
10. $(F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)$ by A2,
11. $(F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow -\neg F_2)$ by 9, 10 and RMP.

This completes the proof.

A simple property following from the definition of proof of rule calculus is so-called compactness as follows.

Proposition 7 (Compactness). *Let* Δ *be a rule base and* R *a rule such that* $\Delta \vdash R$. *There exists a finite subset* Δ' *of* Δ *such that* $\Delta' \vdash R$.

Proposition 8 (Deduction theorem). *Let* Δ *be a rule base and* R_1 *and* R_2 *two rules.* $\Delta \cup \{R_1\} \vdash R_2$ *iff* $\Delta \vdash R_1 \Rightarrow R_2$.

[3] In fact, the Modus Ponens rule in classical logic is, of course, also an inference rule in rule calculus. However, since axiom A1 takes all classical tautologies into account, we can omit the classical Modus Ponens rule here.

Deduction theorem is a very useful tool for proving the consequence relationships in rule calculus. Consider the following example.

Example 3. [Example 1 continued] We prove that $bird \Rightarrow fly \vdash bird \& -\neg fly \Rightarrow fly$. By deduction theorem, we only need to prove $\{bird \Rightarrow fly, bird \; \& \; -\neg fly\} \vdash fly$. This is quite simple from A6 and RMP.

Proposition 9. $\vdash (R_1 \Rightarrow R_2) \Rightarrow (-R_2 \Rightarrow -R_1)$.

Proof. We construct a proof of $-R_1$ from $\{R_1 \Rightarrow R_2, -R_2\}$.

1. $-R_2 \Rightarrow (R_1 \Rightarrow -R_2)$ by A3,
2. $-R_2$ by premises,
3. $R_1 \Rightarrow -R_2$ by 1, 2 and RMP,
4. $(R_1 \Rightarrow R_2) \Rightarrow ((R_1 \Rightarrow -R_2) \Rightarrow -R_1)$ by A9,
5. $R_1 \Rightarrow R_2$ by premises,
6. $(R_1 \Rightarrow -R_2) \Rightarrow -R_1$ by 4, 5 and RMP,
7. $-R_1$ by 3, 6 and RMP.

Thus, $\{R_1 \Rightarrow R_2, -R_2\} \vdash -R_1$. By deduction theorem, $\{R_1 \Rightarrow R_2\} \vdash -R_2 \Rightarrow -R_1$. Again, by deduction theorem, $\vdash (R_1 \Rightarrow R_2) \Rightarrow (-R_2 \Rightarrow -R_1)$.

Proposition 10. *Let F, G and Q be three facts.*

1. $\vdash F \Rightarrow -\neg F$.
2. $\vdash (F \; \& \; G) \Leftrightarrow (F \wedge G)$.
3. $\vdash (-F \mid -G) \Leftrightarrow (\neg F \vee \neg G)$.
4. $\vdash (F \wedge Q \rightarrow G) \Rightarrow (F \; \& \; -G \Rightarrow -Q)$.
5. $\vdash (F \rightarrow G) \Rightarrow (-G \Rightarrow -F)$.

Theorem 1 (Soundness and completeness). *Let R be a rule. R is a rule tautology iff R is a rule theorem. That is, $\models R$ iff $\vdash R$.*

Proof. "soundness:" We first show that all instances of axioms are rule tautologies. As an example, we only present the proofs of A2 and A10 here. Let $\langle T_1, T_2\rangle$ be an HT-interpretation other than $\langle \bot, \bot \rangle$.

A2 Assume that $\langle T_1, T_2\rangle$ is not an HT-model of $(F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)$. Then, there are two cases. Case 1: $\langle T_1, T_2\rangle \models F_1 \rightarrow F_2$ and $\langle T_1, T_2\rangle \not\models F_1 \Rightarrow F_2$. That is, $T_1 \models F_1 \rightarrow F_2$ and a) $\langle T_1, T_2\rangle \models F_1$ and $\langle T_1, T_2\rangle \not\models F_2$ or b) $T_2 \not\models F_1 \Rightarrow F_2$. Thus, $T_1 \models F_1 \rightarrow F_2$ and a) $T_1 \models F_1$ and $T_1 \not\models F_2$ or b) $T_2 \models F_1$ and $T_2 \not\models F_2$. Whichever the case is, it leads to a contradiction. Case 2: $T_2 \not\models (F_1 \rightarrow F_2) \Rightarrow (F_1 \Rightarrow F_2)$. Then, $T_2 \models F_1 \rightarrow F_2$ and $T_2 \not\models F_1 \Rightarrow F_2$. That is, $T_2 \models F_1 \rightarrow F_2$ and $T_2 \models F_1$ and $T_2 \not\models F_2$, a contradiction.

A10 Assume that $\langle T_1, T_2\rangle$ is not an HT-model of $R_1 \mid (R_1 \Rightarrow R_2) \mid -R_2$. Then, $\langle T_1, T_2\rangle \not\models R_1$ and $\langle T_1, T_2\rangle \not\models R_1 \Rightarrow R_2$. Therefore $T_2 \not\models R_1 \Rightarrow R_2$. Thus, $T_2 \not\models R_2$. However, $\langle T_1, T_2\rangle \not\models -R_2$. Thus, By Proposition 1, $T_2 \not\models -R_2$, a contradiction.

We then show that all inferences rules preserve rule tautologies. Suppose that both R_1 and $R_1 \Rightarrow R_2$ are rule tautologies. Given an HT-interpretation $\langle T_1, T_2\rangle$, we have that $\langle T_1, T_2\rangle \models R_1$ and $\langle T_1, T_2\rangle \models R_1 \Rightarrow R_2$. Thus, $\langle T_1, T_2\rangle \models R_2$. This shows that R_2 is also a rule tautology. Hence, soundness holds.

”completeness:” As recently shown in [8], each formula in the logic of here-and-there is equivalent to a set of formulas of the following form:

$$p_1 \wedge \ldots \wedge p_n \wedge \neg p_{n+1} \wedge \ldots \wedge \neg p_m \rightarrow p_{m+1} \vee \ldots \vee p_k \vee \neg p_{k+1} \vee \ldots \vee \neg p_l,$$

where $p_i, (1 \leq i \leq l)$ are atoms. A similar result for rule calculus can be proved in the same way. That is, each rule is equivalent to a set of rules of the following form:

$$F_1 \,\&\, \ldots \,\&\, F_n \,\&\, -F_{n+1} \,\&\, \ldots \,\&\, -F_m \Rightarrow F_{m+1} \,|\, \ldots \,|\, F_k \,|\, -F_{k+1} \,|\, \ldots \,|\, -F_l, \quad (1)$$

where $F_i, (1 \leq i \leq l)$ are facts.

Thus, we only need to prove that for each rule R of form (1), if $\models R$, then $\vdash R$. For convenience, we assume that

$$R = F_1 \;\&\; \ldots \;\&\; F_n \;\&\; -G_1 \;\&\; \ldots \;\&\; -G_m \Rightarrow P_1 \;|\; \ldots \;|\; P_k \;|\; -Q_1 \;|\; \ldots \;|\; -Q_l.$$

Let $F = \bigwedge_{1 \leq i \leq n} F_i$ and $Q = \bigwedge_{1 \leq i \leq l} Q_i$ Then, one of the following statements must hold.

1. There exists $i, (1 \leq i \leq k)$ such that $F \rightarrow P_i$ is a classical tautology.
2. $F \rightarrow \neg Q$ is a classical tautology.
3. There exists $j, (1 \leq j \leq m)$ such that $(F \wedge Q) \rightarrow G_j$ is a classical tautology.

Suppose otherwise, then there exists a propositional assignment π_0 such that $\pi_0 \models F \wedge Q$; there exists a propositional assignment $\pi_i, (1 \leq i \leq k)$ such that $\pi_i \models F \wedge \neg P_i$; there exists a propositional assignment $\pi'_j, (1 \leq i \leq m)$ such that $\pi'_j \models F \wedge Q \wedge \neg G_j$. Let T_1 be the theory such that the set of its models is $\{\pi_0, \pi_i, \pi'_j, (1 \leq i \leq k), (1 \leq j \leq l)\}$, and T_2 be the theory such that the set of its models is $\{\pi'_j, (1 \leq j \leq l)\}$. It is easy to check that $\langle T_1, T_2\rangle$ is not an HT-model of R, a contradiction.

Thus, one of the three previous statements holds. Hence, R can be proved according to axioms and Proposition 6, Proposition 9 and Proposition 10. As an example, we prove the third case. Without loss of generality, suppose that $(F \wedge Q) \rightarrow G_1$ is a classical tautology. Then, by point 4 in Proposition 10, $F \;\&\; -G_1 \Rightarrow -Q$ is a rule tautology. Then, by point 2 and point 3 in Proposition 10, $F_1 \& \ldots \& F_n \& -G_1 \Rightarrow -Q_1 | \ldots | -Q_l$ is a rule tautology. By A6 and A7, R is a rule tautology.

From compactness, deduction theorem and soundness and completeness, we have the following result.

Corollary 2. *Let Δ be a rule base and R a rule. R is implied by Δ iff R is a consequence of Δ. That is, $\Delta \models R$ iff $\Delta \vdash R$.*

4 Other properties

In this section, we discuss some other important properties of rule calculus.

From Proposition 10, one may claim that rule connectives and corresponding classical connectives play the same roles to some extent. However, this is not the case. For instance, $\nvdash (F|G) \Leftrightarrow (F \vee G)$, where F and G are two facts. As another example, from A2, $\vdash (F \rightarrow G) \Rightarrow (F \Rightarrow G)$. However, $\nvdash (F \Rightarrow G) \Rightarrow (F \rightarrow G)$. Consequently, $\nvdash -F \Rightarrow \neg F$ although $\vdash \neg F \Rightarrow -F$.

In fact, rule calculus is more like the intermediate logic G3. It is clear that the former is an extension of the latter. Hence, theorems not in G3 are not rule theorems in rule calculus. For example, $R \mid -R$ is not a rule theorem. On the other hand, theorems in G3 can be extended for rule calculus. For example, the following property holds.

Proposition 11. *Let R_1, R_2, R_3 and R_4 be four rules.*

1. $\{R_1 \Rightarrow R_2, R_2 \Rightarrow R_3\} \vdash R_1 \Rightarrow R_3$.
2. $\{R_1 \Rightarrow R_2, R_3 \Rightarrow R_4\} \vdash R_1 \mid R_3 \Rightarrow R_2 \mid R_4$.
3. $R_1 \Rightarrow - - R_1$.

Proposition 12. *Let R be a theorem in G3 composed from a set of atoms $P = \{p_1, \dots, p_n\}$ and $\Delta = \{R_i, (1 \leq i \leq n)\}$ are n rules associated with each p_i. Then, $R(P/\Delta)$ is a rule theorem, where $R(P/\Delta)$ is the rule obtained from R by replacing every occurrence of $p_i, (1 \leq i \leq n)$ with corresponding R_i simultaneously.*

Proposition 13. *Let F_1 and F_2 be two facts. $F_1 \mid F_2 \vdash F_1 \vee F_2$.*

However, $F_1 \vee F_2 \nvdash F_1 \mid F_2$. For example, $\langle\{F_1 \vee F_2\}, \{F_1 \vee F_2\}\rangle$ is an HT-model of $F_1 \vee F_2$ but not an HT-model of $F_1 \mid F_2$.

Proposition 14 (Substitution). *Let R be a rule theorem and R_1 and R_2 two rules. $R(R_1/R_2)$, the rule obtained from R by replacing every occurrence of R_1 with R_2 simultaneously, is a rule theorem as well.*

Proposition 15. *Let F_1, F_2 and F_3 be three facts. $\{F_1 \Rightarrow F_2, F_2 \,\&\, -\neg F_3 \Rightarrow F_3\} \vdash F_1 \,\&\, -\neg F_3 \Rightarrow F_3$.*

However, $\{F_2 \Rightarrow F_3, F_1 \& -\neg F_2 \Rightarrow F_2\} \nvdash F_1 \& -\neg F_3 \Rightarrow F_3$ since $\langle\{p_1\}, \{p_1 \wedge \neg p_2\}\rangle$ is an HT-models of $\{p_2 \Rightarrow p_3, p_1 \& -\neg p_2 \Rightarrow p_2\}$ but not an HT-models of $p_1 \& -\neg p_3 \Rightarrow p_3$. In addition, as shown in Example 2, $\{F_1 \,\&\, -\neg F_2 \Rightarrow F_2, F_2 \,\&\, -\neg F_3 \Rightarrow F_3\} \nvdash F_1 \,\&\, -\neg F_3 \Rightarrow F_3$.

Interestingly, rule calculus is monotonic although the extension semantics of default logic is dealing with nonmonotonicity.

Proposition 16 (Monotonicity). *Let R be a rule and Δ and Δ' are two rule bases such that $\Delta \subseteq \Delta'$. If $\Delta \vdash R$, then $\Delta' \vdash R$.*

Theorem 2 (Complexity). *Checking whether a rule R has at least one HT-model is NP complete.*

Proof. Hardness is obvious since a fact is satisfiable iff it has at least one HT-model. For membership, we first prove a lemma by induction. Given two HT-interpretations $\langle T_1, T_2\rangle$ and $\langle T_1', T_2'\rangle$ and a set of facts Γ, if for all $F \in \Gamma$, $T_1 \models F$ iff $T_1' \models F$, and so do T_2 and T_2', then for all rules R composed from Γ and rule connectives, $\langle T_1, T_2\rangle \models R$ iff $\langle T_1', T_2'\rangle \models R$.

Suppose that R is composed from the set of facts $\Gamma = \{F_1, \ldots, F_n\}$ and $\langle T_1, T_2\rangle \models R$. Without loss of generality, suppose that $T_1 \models F_i, (1 \leq i \leq m)$ and $T_1 \not\models F_i, (m < i \leq n)$. Therefore, there exists a propositional assignment $\pi_i, (m < i \leq n)$ such that $\pi_i \models \bigwedge_{1\leq j\leq m} F_j \wedge \neg F_i$. Let T_1' be the theory such that its models are $\pi_i, (m < i \leq n)$. It is easy to see that T_1 and T_1' agree the same on Γ. We can construct T_2' in the same way. We have that $T_1' \subseteq T_2'$. Therefore $\langle T_1', T_2'\rangle \models R$. This shows that if a rule R has a model, then it has a model which can be represented polynomially. It follows that checking whether a rule R has at least one HT-model is in NP.

It is well known that most of the decision problems in default logics lie on the second level of polynomial hierarchy [6], even restricted to some special subclasses [5]. Surprisingly, although rule calculus seems more complicated than others such as skeptical and credulous reasoning, its complexity is lower than them according to Theorem 2. This draws an opposite conclusion. That is, the problem of rule calculus is, indeed, simpler than other reasoning tasks of default logic. However, this does not mean that the former is weaker than the latter since they are dealing with different reasoning tasks of default rules.

5 Applications

Since the notion of rule deduction is an extension of deduction in propositional calculus, many propositional logic based deductive reasoning tasks can be lifted to corresponding cases in rule calculus. In this section, we briefly discuss three applications, which seem to be hard to deal with in default logic on their own. However, by using rule calculus, these problems can be easily solved. Due to a space limit, we only outline the basic ideas here.

Irrelevance in default logic

As pointed by Lang et al. [13], *irrelevance* is an important notion in propositional logic. According to Lang et al.'s definition, a propositional formula F is *irrelevant* to a set V of atoms iff there exists another formula G such that F is equivalent to G and $Atom(G) \cap V = \emptyset$, where $Atom(G)$ is the set of atoms appeared in G. The notion of irrelevance in rule calculus can be defined in a similar way. That is, a rule R_1 is *irrelevant* to a set V of atoms iff there exists another rule R_2 such that $\models R_1 \Leftrightarrow R_2$ and $Atom(R_2) \cap V = \emptyset$, where $Atom(R_2)$ is the set of atoms occurred in R_2.

Having defined the notion of irrelevance in rule calculus, we can define other related notions such as forgetting in a similar way as shown in [13].

Generality among default rules

The concept of generality is a foundational basis of inductive logic programming [14, 15] - a subfield of machine learning and has been successfully applied to some real domains. Inductive logic programming started from propositional logic [14] but then has focused on Horn clauses (namely logic programs) [15].

In propositional logic, the generality relationships between two formulas can be defined in a simple way as shown in [14]. That is, a formula F_1 is said to be *more general* than a formula F_2 iff $F_1 \models F_2$ and $F_2 \not\models F_1$. We can lift this notion to the case between two rules. A rule R_1 is said to be *more general* than a formula R_2 iff $R_1 \models R_2$ and $R_2 \not\models R_1$. Moreover, this notion can be easily extended to the cases with a background rule base. A rule R_1 is said to be *more general* than a rule R_2 *relative to* a rule base Δ iff $\Delta \cup \{R_1\} \models R_2$ and $\Delta \cup \{R_2\} \not\models R_1$. It is obvious that this definition is a generalization of the definition of generality in propositional calculus since all propositional formulas are also rules.

Inoue and Sakama [16] introduced several kinds of generality relationship between default theories. Although their definitions are based on disjunctive default logic, these can be easily extended to general default logic. We write $Ext(R)$ to denote the set of extensions of a rule R. Let R_1 and R_2 be two rules, according to Inoue and Sakama's definitions, R_1 is said to be *more $\sharp$-general* than R_2 iff for all $T_1 \in Ext(R_1)$, there exists $T_2 \in Ext(R_2)$ such that $T_2 \subseteq T_1$; R_1 is said to be *more $\flat$-general* than R_2 iff for all $T_2 \in Ext(R_2)$, there exists $T_1 \in Ext(R_1)$ such that $T_2 \subseteq T_1$; R_1 is said to be *strongly more $\sharp$-general* than R_2 iff for all rules R_3 $R_1 \,\&\, R_3$ is more $\sharp$-general than $R_2 \,\&\, R_3$; R_1 is said to be *strongly more $\flat$-general* than R_2 iff for all rules R_3 $R_1 \,\&\, R_3$ is more $\flat$-general than $R_2 \,\&\, R_3$. However, the cases relative to a background default theory were not considered in their approach.

Our definition of generality does not coincide with any of these notions. For instance, let p_1 and p_2 be two atoms. We have that $\top$ is both $\sharp$-general and $\flat$-general than $-p_1$ but the former is not more general than the latter in our definition. Meanwhile, $p_1 \wedge p_2$ is more general than p_1 in our definition but the former is neither strongly more $\sharp$-general nor strongly $\flat$-general than the latter. One major difference between these two approaches is that Inoue and Sakama's notions are defined based on the sets of extensions of default theories. However, two rules sharing the same set of extensions may play completely different roles in rule calculus.

Revising default rule bases

Belief revision has been an important topic in solving information conflict in reasoning about agents. In most existing approaches and systems, an agent's knowledge base is usually represented by a set of classical propositional formulas, then various revision methods have been developed by researchers to solve the inconsistency by revising a knowledge base by a new piece of information.

Under the framework of rule calculus, this work can be generalized to nonmonotonic knowledge base revision. That is, in our setting, each agent's knowledge is represented as a rule base, and the problem is how to revise this rule base by giving a new default rule.

We may specify a formulation of rule base revision by generalizing approaches for propositional belief revision, for instance, the WIDTIO approach [17]. Given a rule base Δ and a rule R, we say that Δ' is a maximal subset of Δ consistent with R iff a) $\Delta' \subseteq \Delta$, b) $\Delta' \cup \{R\}$ is not a rule contradiction, and c) there does not exist Δ''

satisfying the above two conditions and $\Delta' \subset \Delta'' \subseteq \Delta$. The rule base revision operator $\circ$ is defined as $\Delta \circ R = \bigcap \Delta' \cup \{R\}$. This revision operator also satisfies the well-known AGM postulates.

6 Conclusion

In this paper, we extend the logic of here-and-there as a general semantics for default rules. Meanwhile, we propose a corresponding axiom system for rule calculus and prove the soundness and completeness theorem (see Theorem 1). We also discuss other properties in rule calculus, including complexity issues (see Theorem 2).

The notion of strong equivalence in default logic can be directly captured in rule calculus (See Corollary 1). Corollary 1 also indicates the intuition behind rule deduction, that is, a rule R is a consequence of a rule base Δ means that R provides no more information by giving Δ. In other words, R can be eliminated by giving Δ. On the other hand, given the fact that answer set programming is a special case of default logic, our approach also shows that the logic of here-and-there and its axiomatic counterpart G3 can capture the consequence relationships among answer set programs. In fact, restricted to answer set programs (i.e., facts are atoms instead of arbitrary propositional formulas), rule calculus coincides with the notion of SE-consequence [18, 19].

Rule calculus is an extension of propositional calculus and also an extension of the intermediate logic G3 [7] in the sense that the connectives in G3 are represented as rule connectives. It can also be considered as an extended logic of formalizing normality [20] since the sentence "A normally implies B" can be represented as $A \,\&\, -\neg B \Rightarrow B$ as suggested by Reiter [4].

Rule calculus is different from conditional logic [21] although both of them introduce new connectives into propositional calculus. There are two syntactic differences. First, conditional logic only introduces a conditional connective $>$. Second, whereas conditional logic allows arbitrary compositions of atoms and connectives including $>$, in most cases, it uses $>$ as a lower level connective, whilst in rule calculus, classical connectives are at the lower level. Certainly, the axiom systems and semantics of these two logics are basically dissimilar. For instance, the conditional connective $>$ is intuitively stronger than $\rightarrow$ in conditional logic, whilst the rule implication $\Rightarrow$ is, to some extent, weaker than $\rightarrow$ in rule calculus.

Another related work is so-called proof theory of default logic [22, 23], which aims to define a proof-theoretical system for determining whether a propositional formula is in all (or some) extensions of a default theory. It differs from rule calculus in several aspects. Firstly, proof theory of default logic is operating on the level of extension semantics, whilst rule calculus is focused on the here-and-there semantics. Secondly, The consequence concerned in rule calculus is, in general, default rules instead of propositional formulas. Finally, even restricted to the cases of facts, as we mentioned earlier (See Proposition 2), these two systems do not coincide with each other.

Acknowledgement

This work is supported by Australian Research Council (ARC) Discovery Projects grant DP0666540. The authors thank the anonymous reviewers for their valuable comments.

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