Knowledge Forgetting: Properties and Applications

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Abstract

In this paper we study a formal notion of knowledge forgetting in S5 modal logic. We propose four postulates and prove that these postulates precisely characterize both semantic and logical properties of knowledge forgetting. We then investigate possible applications of knowledge forgetting in various epistemic reasoning scenarios. In particular, we show that different forms of knowledge updates may be represented via knowledge forgetting. We also demonstrate how knowledge forgetting can be used in formalizing and reasoning about knowledge games with bounded memory.

Key words: epistemic reasoning, reasoning about belief and knowledge, knowledge update, knowledge games, nonmonotonic reasoning

1 Introduction

Epistemic reasoning concerns the problem of how to reason about agents' epistemic states (knowledge) in a dynamic environment, e.g. [6,26]. In the last decade, it has been demonstrated that epistemic reasoning has many important applications in computer science and AI [23,25]. Amongst various theories and approaches, one major assumption in the study of epistemic reasoning is that agents always remember their previous knowledge (i.e. agents have perfect recall) [11,24]. However, as pointed by Fagin *et al.*: "*There are often scenarios of interest where we want to model the fact that certain information is discarded. In practice, for example, an agent may simply not have enough memory capacity to remember everything he has learned.*" [11, page 129]. Hence knowledge forgetting is an important behaviour for an agent under certain circumstances.

As a logical notion, forgetting was first studied in propositional and first order logics from a KR perspective by Lin and Reiter [21]. Over the years, researchers have used the notion of propositional forgetting to deal with issues in abductive reasoning, belief revision/update, and reasoning about knowledge [19,22,27]. In recent years, theories of forgetting have also been proposed under the answer set programming semantics and used in solving logic program conflicts and updates [10,28]. We can see that forgetting has many applications in knowledge representation and reasoning.

However, existing forgetting definitions in propositional logic and answer set programming are not directly applicable in modal logics. For instance, in propositional forgetting theory, forgetting atom q from $T \equiv (p \rightarrow q) \wedge ((q \wedge r) \rightarrow s)$ is equivalent to a formula $T[q/\top] \vee T[q/\bot]$, where $T[q/\top]$ is a formula obtained from T by replacing each q with $\top$ and $T[q/\bot]$ is obtained from T by replacing each q with $\bot$, which is $(r \rightarrow s) \vee \neg p$. However, this method cannot be extended to an S5 modal logic formula. Consider an S5 formula $T' \equiv \neg Kq \wedge \neg K \neg q$. If we want to forget atom q from T' by using the above method, we would have $T'[q/\top] \vee T'[q/\bot] \equiv \bot$. This is obviously not correct because after forgetting q, the agent's knowledge set should not become inconsistent!

An earlier formal study on the concept of forgetting in epistemic logic (called knowledge forgetting) was Baral and Zhang's work on knowledge update [1], where they treated knowledge forgetting as a special form of update with the effect $\neg K\phi \wedge \neg K \neg \phi$: after knowledge forgetting a propositional formula ϕ, the agent would neither know ϕ nor $\neg \phi$.

Recently, van Ditmarsch *et al.* have also considered the issue of forgetting in a modal logical context [8]. Their forgetting concept is similar to Baral and Zhang's forgetting update in the sense that after (knowledge) forgetting atom p the agent should conclude $\neg Kp \wedge \neg K \neg p$, but based on a dynamic modal logic. They showed that their dynamic modal logic of forgetting is sound and complete.

While both Baral and Zhang's and van Ditmarsch *et al.*'s work have made interesting contributions to formalize the notion of knowledge forgetting, its underlying semantics still remains unclear. For instance, neither of their knowledge forgetting notions always results in intuitive solutions. Specifically, by restricting to classical propositional logic, their forgetting notions are not consistent with propositional variable forgetting.

In recent modal logic research, notions of bisimulation quantification and uniform interpolation have been extensively studied, which eventually provide semantic and logical interpretations for knowledge forgetting. Based on the notion of bisimulation quantification, French, Ghilardi and Zawadowski defined the semantics of formula $\exists V \phi$ [12,14], where V is a finite set of propositional atoms and ϕ is a formula in certain modal logic. Intuitively, an interpretation M is a model of $\exists V \phi$ iff there is a model M' of ϕ and M and M' are *bisimilar* with exception over atoms in V, i.e. M and M' are V-bisimilar.

Ghilardi *et al.* studied the notion of *uniform interpolation* in modal logic, and indicated that S5 has the uniform interpolation property. Informally, this means that for every S5 formula ϕ and every finite set V of atoms, there exists an S5 formula $\exists V\phi$ which does not contain atoms from V but is logically closest to ϕ in some sense [15]. It is not difficult to show that this logical definition of $\exists V\phi$ is equivalent to French's semantic interpretation of $\exists V\phi$ [12].

Nevertheless, two major issues still remain unaddressed. Firstly, it is not clear yet whether we can precisely capture both semantic and logical interpretations of knowledge forgetting through general principles or postulates, as in the analogous work in belief revision and update [17]. Secondly, the application of knowledge forgetting to different epistemic reasoning has not been thoroughly studied. In this paper we propose four general postulates for knowledge forgetting and show that these four postulates precisely characterize the notion of knowledge forgetting described above in S5 modal logic. We then focus on the study of applications of knowledge forgetting in various epistemic reasoning scenarios. Specifically, we study the relationship between knowledge update and knowledge forgetting, and show that knowledge forgetting may be used as a flexible notion to represent different forms of knowledge updates. We also demonstrate how knowledge forgetting can be used in formalizing and reasoning about knowledge games with bounded memory.

The rest of the paper is organized as follows. Section 2 provides some logical background and presents a semantic definition of knowledge forgetting under our context. Section 3 proves a representation theorem for knowledge forgetting and studies other related semantic properties. Section 4 shows that different forms of knowledge updates may be represented through knowledge forgetting. Section 5 demonstrates an interesting application of knowledge forgetting in describing knowledge games with bounded memory. Finally section 6 concludes this paper with some remarks.

2 Defining knowledge forgetting

Our knowledge forgetting will be defined on a basis of propositional modal logic S5. Let $Atom$ be a set of *atoms* (also called *variables*). The language $\mathcal{L}$ of propositional S5 modal logic is defined recursively by $Atom$, classical connectives $\bot$, $\neg$, $\supset$ and a modal operator K as follows:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi \supset \psi \mid K\phi,$$

where $p \in Atom$. $\top$, $\phi \wedge \psi$ and $\phi \vee \psi$, are defined as the standard way. Elements in $\mathcal{L}$ are called *formulas*. Formulas without modal operators are called *objective formulas*. A *knowledge set* is a finite set of formulas. *Literals* are atoms and their negations. Let ϕ be a formula and Γ a knowledge set, we write $Var(\phi)$ and $Var(\Gamma)$ to denote the sets of atoms occurred in ϕ and Γ respectively.

For convenience, we usually use $a, b, c, \cdots, p, q, \cdots$ to denote atoms; $\phi, \psi, \upsilon, \cdots$ to denote formulas; and Γ, T, $\cdots$ to denote knowledge sets. Sometimes, we also use the conjunction $\phi_1 \wedge \cdots \wedge \phi_n$ to represent a finite set of formulas $\{\phi_1, \cdots, \phi_n\}$.

A *Kripke structure* is a triple $S = \langle W, R, L\rangle$, where W is a set of possible worlds, R an equivalence relation on W, and L a set of interpretations for each world in W [1] . A *Kripke interpretation* is a pair $M = \langle S, w\rangle$ where $w \in W$. As mentioned in [23], an S5 Kripke interpretation can be simplified as $M = \langle W, w\rangle$, where W is the set of all possible worlds, each world is identified as a set of atoms, and $w \in W$ is called the *actual world*. In this case, we call $M = \langle W, w\rangle$ a k*-interpretation*.

The *satisfaction relation* $\models$ between k-interpretations and formulas in $\mathcal{L}$ is defined recursively as follows [2] :

(1) $\langle W, w\rangle \not\models \bot$;
(2) $\langle W, w\rangle \models p$ iff $p \in w$, where $p \in Atom$;
(3) $\langle W, w\rangle \models \neg\phi$ iff $\langle W, w\rangle \not\models \phi$;
(4) $\langle W, w\rangle \models \phi \supset \psi$ iff $\langle W, w\rangle \not\models \phi$ or $\langle W, w\rangle \models \psi$;
(5) $\langle W, w\rangle \models K\phi$ iff $\forall w' \in W$, $\langle W, w'\rangle \models \phi$.

We say that M is a k*-model* of ϕ iff $M \models \phi$. An S5 formula ϕ is *satisfiable* if it has a k-model. ϕ is *valid* if for each k-interpretation $\langle W, w\rangle$, $\langle W, w\rangle \models \phi$. In this case, we also denote $\models \phi$. We write $Mod(\phi)$ (or $Mod(\Gamma)$ if Γ is a finite set of formulas) to denote the set of all k-models of ϕ (or Γ resp.). We say that two S5 formulas (knowledge sets) ϕ and ψ are *equivalent*, denoted by $\phi \equiv \psi$, iff $Mod(\phi) = Mod(\psi)$. We denote $\phi \models \psi$ iff $Mod(\phi) \subseteq Mod(\psi)$. That is, we treat $\models$ as the local consequence relation in S5.

To present a formal definition of knowledge forgetting, we need the concepts of *bisimulation* [2] and *bisimulation quantification* that have been studied in modal logic [12,14]. Let w and w' be two worlds identified as two sets of atoms respectively, and $V \subseteq Atom$ a set of atoms. We denote $w \simeq_V w'$ if for all $p \in Atom \setminus V$, $p \in w$ iff $p \in w'$.

Definition 1 *[12] Let $M = \langle W, w\rangle$ and $M' = \langle W', w'\rangle$ be two k-interpretations, and $V \subseteq Atom$ a set of atoms. We say that M and M' are V-*bisimilar *(i.e. bisimilar with exception on V), denoted by $M \leftrightarrow_V M'$, iff the following conditions hold:*

(1) $w \simeq_V w'$;
(2) $\forall w^* \in W$, $\exists w^{*'} \in W'$ such that $w^* \simeq_V w^{*'}$ (the forth condition); and
(3) $\forall w^{*'} \in W'$, $\exists w^* \in W$ such that $w^* \simeq_V w^{*'}$ (the back condition).

[1] In this paper, we will only consider Kripke structures for logic S5 where R is restricted to be an equivalence relation. In the rest of the paper, we will not explicitly mention this whenever there is no confusion.

[2] We write $\langle W, w\rangle \not\models F$ if it is not the case that $\langle W, w\rangle \models F$.

Note that even if $M \leftrightarrow_V M'$, M and M' may have different number of worlds.

Proposition 1 *Let M_1 and M_2 be two k-interpretations, V a set of atoms and p an atom. If M_1 and M_2 are $V \cup \{p\}$-bisimilar, then there exists a k-interpretation M' such that M_1 and M' are $\{p\}$-bisimilar and M' and M_2 are V-bisimilar.*

Proof. Let $M_1 = \langle W_1, s_1 \rangle$ and $M_2 = \langle W_2, s_2 \rangle$ and $M_1 \simeq_{V \cup \{p\}} M_2$. We construct a k-interpretation $M' = \langle W', s' \rangle$ as follows: (1) for all pairs $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 \simeq_V w_2$, let $w' \in W'$ and (a) $p \in w'$ iff $p \in w_2$, (b) for all atoms $q \in V$, $q \in w'$ iff $q \in w_1$, (c) for all other atoms q', $q' \in w'$ iff $q' \in w_2$ iff $q \in w_1$; (2) delete duplicated worlds in W'; (3) let s' be the world such that (a) $p \in s'$ iff $p \in s_2$, (b) for all atoms $q \in V$, $q \in s'$ iff $q \in s_1$, (c) for all other atoms q', $q' \in s'$ iff $q' \in s_2$ iff $q' \in s_1$. Then it is easy to verify that $M_1 \leftrightarrow_{\{p\}} M'$ and $M' \leftrightarrow_V M_2$. $\Box$

The following are also standard results from bisimulation quantification [12].

Proposition 2 *The relation $\leftrightarrow_V$ is an equivalence relation.*

Proposition 3 *If two k-interpretations M_1 and M_2 are V-bisimilar, then they satisfy the same formulas if the latter do not contain atoms from V.*

Now we define knowledge forgetting as follows.

Definition 2 *(***Knowledge forgetting***) Let Γ be a knowledge set and $V \subseteq Atom$ a set of atoms. A knowledge set, denoted as $\mathsf{KForget}(\Gamma, V)$, is the result of* knowledge forgetting *V from Γ, if the following condition holds:*

$$Mod(\mathsf{KForget}(\Gamma, V)) = \{M' \mid \exists M \in Mod(\Gamma) \text{ and } M \leftrightarrow_V M'\}.$$

From Definition 2, we can see that for each k-model M of Γ, M is a k-model of $\mathsf{KForget}(\Gamma, V)$. Further, if we simply add all worlds into M that differ from worlds of M only on the evaluation of variables in V, then the newly formed k-interpretation M' is also a k-model of $\mathsf{KForget}(\Gamma, V)$.

Example 1 *Consider knowledge sets $K(p \vee q)$, $Kp \vee Kq$ and $K(p \wedge q)$. From Definition 2, it is easy to check that $\mathsf{KForget}(K(p \vee q), \{p\}) \equiv \top$, $\mathsf{KForget}(Kp \vee Kq, \{p\}) \equiv \top$, and $\mathsf{KForget}(K(p \wedge q), \{p\}) \equiv Kq$.*

We should mention that our knowledge forgetting definition $\mathsf{KForget}(\Gamma, V)$ is equivalent to the semantic definition of formula $\exists V \Gamma$ in [12]. It is also equivalent to the notion of uniform interpolation - a logical definition of formula $\exists V \Gamma$ [15]. Ghilardi *et al.* further described an algorithm to construct formula $\exists V \Gamma$ (or $\mathsf{KForget}(\Gamma, V)$ in our context) with exponential size of Γ.

3 Semantic characterizations

In this section we study essential semantic properties of knowledge forgetting which have not been addressed in previous work. We will first propose a set of postulates and show that these postulates precisely characterize the semantics of knowledge forgetting. We then discuss other semantic properties of knowledge forgetting. Related computational properties of knowledge forgetting have been studied in [29].

3.1 A representation theorem

Consider a formula ϕ. Intuitively, if a propositional variable a does not occur in $Var(\phi)$, we may consider that ϕ is *irrelevant* to a. It is not surprising that the notion of irrelevance plays an important role in characterizing the semantics of knowledge forgetting. We first give the following formal definition.

Definition 3 *(***Irrelevance***) Let* Γ *be a knowledge set and* V *a set of atoms. We say that* Γ *is* irrelevant *to* V*, denoted by* $IR(\Gamma, V)$*, if there exists a knowledge set* Γ' *such that* $\Gamma \equiv \Gamma'$ *and* $Var(\Gamma') \cap V = \emptyset$*.*

Let Γ and Γ' be two knowledge sets and V a set of atoms. Now we propose the following postulates:

(W) Weakening: $\Gamma \models \Gamma'$.
(PP) Positive Persistence: if $IR(\phi, V)$ and $\Gamma \models \phi$, then $\Gamma' \models \phi$.
(NP) Negative Persistence: if $IR(\phi, V)$ and $\Gamma \not\models \phi$, then $\Gamma' \not\models \phi$.
(IR) Irrelevance: $IR(\Gamma', V)$.

By specifying $\Gamma' \equiv \mathsf{KForget}(\Gamma, V)$, **(W), (PP), (NP)** and **(IR)** are called *postulates for knowledge forgetting*. Let us take a closer look at these postulates. **(W)** is an essential requirement for knowledge forgetting: after forgetting some information from a knowledge set, the resulting knowledge set then becomes weaker. Indeed, as demonstrated in propositional variable forgetting [21,22], forgetting weakens the original formula. The postulates of positive persistence **(PP)** and negative persistence **(NP)** simply state that knowledge forgetting a set of atoms should not affect those positive or negative information respectively that is irrelevant to this set of atoms. Finally, irrelevance **(IR)** means that after knowledge forgetting, the resulting knowledge set should be irrelevant to those atoms which we have (knowledge) forgotten. We argue that these postulates capture the basic properties that knowledge forgetting should satisfy.

Now we have the following representation theorem which states that our forgetting postulates indeed precisely characterize the underlying knowledge forgetting semantics.

Theorem 1 *(***Representation theorem***) Let* Γ *and* Γ' *be two knowledge sets and*

$V \subseteq Atom$ *a set of atoms. Then the following statements are equivalent:*

(1) $\Gamma' \equiv \mathsf{KForget}(\Gamma, V)$*;*
(2) $\Gamma' \equiv \{\phi \mid \Gamma \models \phi, IR(\phi, V)\}$*;*
*(3) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.*

Proof. It is observed that $1 \Leftrightarrow 2$ is implied by the result that S5 has uniform interpolation property, mentioned in Ghilardi *et al.*'s work [15]. $2 \Rightarrow 3$ is obvious from Definitions 2 and 3. Now we show $3 \Rightarrow 2$. Suppose that all postulates hold. By Positive Persistence, we have $Mod(\Gamma') \subseteq Mod(\{\phi \mid \Gamma \models \phi, IR(\phi, V)\})$. On the other hand, by postulate **(IR)**, we know that Γ' is irrelevant to V. Also note that Γ' is a finite set of formulas. Further from postulate **(W)**, we have $\Gamma \models \Gamma'$. Therefore, $\Gamma' \in \{\phi \mid \Gamma \models \phi, IR(\phi, V)\}$. So $Mod(\{\phi \mid \Gamma \models \phi, IR(\phi, V)\}) \subseteq Mod(\Gamma')$ holds. $\square$

Theorem 1 is significant in the sense that it provides an "if and only if" characterization on knowledge forgetting. That is, given a knowledge set Γ and a set of atoms V, an S5 formula Γ' represents a result of knowledge forgetting V from Γ if Γ' satisfies postulates (**W**), (**PP**), (**NP**) and (**IR**), and *vice versa*.

Corollary 1 *Let* Γ *be a knowledge base,* ϕ *a formula and* V *a set of atoms. If* $IR(\phi, V)$*, then* $\mathsf{KForget}(\Gamma, V) \models \phi$ *iff* $\Gamma \models \phi$*.*

3.2 Other semantic properties

As we mentioned in Introduction, the notion of forgetting has been defined and used in a variety of contexts under propositional logic [18,19,21]. It is important to know the relationship between variable forgetting in propositional logic and knowledge forgetting in S5 propositional modal logic.

Let ϕ be an objective formula. We use $\phi[p/\bot]$ and $\phi[p/\top]$ to denote the formulas obtained from ϕ by replacing atom p with $\bot$ and $\top$ respectively. Then formula $\mathsf{Forget}(\phi, p)$ is obtained from ϕ by *forgetting* p from ϕ, if $\mathsf{Forget}(\phi, p) \equiv \phi[p/\bot] \vee \phi[p/\top]$. By forgetting a set of atoms V in ϕ, we recursively define $\mathsf{Forget}(\phi, V \cup \{p\}) = \mathsf{Forget}(\mathsf{Forget}(\phi, p), V)$, where $\mathsf{Forget}(\phi, \emptyset) = \phi$. We first have the following result.

Theorem 2 *Let* ϕ *be an objective formula and* V *a set of atoms. Then we have* $\mathsf{KForget}(\phi, V) \equiv \mathsf{Forget}(\phi, V)$ *and* $\mathsf{KForget}(K\phi, V) \equiv K(\mathsf{Forget}(\phi, V))$.

Proof. We prove Result 1 as follows. Suppose that $M = \langle W, w \rangle$ is a k-model of $\mathsf{KForget}(\phi, V)$. Then there exists $M' = \langle W', w' \rangle \in Mod(\phi)$ such that $M' \leftrightarrow_V M$. Thus, $w' \simeq_V w$. Since $w' \models \phi$, $w \models \mathsf{Forget}(\phi, V)$. Hence, M is also a k-model of $\mathsf{Forget}(\phi, V)$. On the other hand, suppose that $M = \langle W, w \rangle$ is a k-model of $\mathsf{Forget}(\phi, V)$. Then there exists $w' \models \phi$ and $w' \simeq_V w$. Construct a k-interpretation

$M' = \langle W', w' \rangle$ such that $W' = W \cup \{w'\} \backslash \{w\}$. It's clear that M' is a k-model of ϕ and $M' \leftrightarrow_V M$. Hence, M is also a k-model of $\mathsf{KForget}(\phi, V)$.

Now consider Result 2. Suppose that $M = \langle W, w \rangle$ is a k-model of $\mathsf{KForget}(K\phi, V)$. Then there exists $M' = \langle W', w' \rangle \in Mod(K\phi)$ such that $M' \leftrightarrow_V M$. We have that for all $w'_1 \in W'$, $w'_1 \models \phi$. Since $M' \leftrightarrow_V M$, for all $w_1 \in W$, there exists $w'_1 \in W'$ such that $w_1 \simeq_V w'_1$. Therefore $w_1 \models \mathsf{Forget}(\phi, V)$. This shows that $M \models K(\mathsf{Forget}(\phi, V))$. On the other hand, suppose that $M = \langle W, w \rangle$ is a k-model of $K(\mathsf{Forget}(\phi, V))$. Then for all $w_1 \in W$, $w_1 \models \mathsf{Forget}(\phi, V)$, and there exists $w'_1 \models \phi$ and $w'_1 \simeq_V w$. Construct a k-interpretation $M' = \langle W', w' \rangle$ such that W' is the set of all w'_1 mentioned above and $w' \simeq_V w$. It's clear that M' is a k-model of $K(\phi)$ and $M' \leftrightarrow_V M$. Hence, M is also a k-model of $\mathsf{KForget}(K\phi, V))$. $\Box$

Theorem 2 simply reveals that propositional variable forgetting is a special case of knowledge forgetting, and also knowledge forgetting of formulas with the form $K\phi$ (where ϕ is objective) can be achieved through the corresponding propositional variable forgetting. The following results further illustrate other essential semantic properties of knowledge forgetting.

Theorem 3 *Let Γ, Γ_1 and Γ_2 be three knowledge sets, ϕ_1 and ϕ_2 two formulas, and V a set of atoms. Then the following results hold:*

(1) $\mathsf{KForget}(\Gamma, V)$ *is satisfiable iff Γ is satisfiable;*
(2) If $\Gamma_1 \equiv \Gamma_2$, then $\mathsf{KForget}(\Gamma_1, V) \equiv \mathsf{KForget}(\Gamma_2, V)$*;*
(3) if $\Gamma_1 \models \Gamma_2$, then $\mathsf{KForget}(\Gamma_1, V) \models \mathsf{KForget}(\Gamma_2, V)$*;*
(4) $\mathsf{KForget}(\phi_1 \vee \phi_2, V) \equiv \mathsf{KForget}(\phi_1, V) \vee \mathsf{KForget}(\phi_2, V)$*;*
(5) $\mathsf{KForget}(\phi_1 \wedge \phi_2, V) \models \mathsf{KForget}(\phi_1, V) \wedge \mathsf{KForget}(\phi_2, V)$.

Proof. To prove Result 1, suppose that M is a k-model of Γ. Then M is also a k-model of $\mathsf{KForget}(\Gamma, V)$. This shows that $\mathsf{KForget}(\Gamma, V)$ is satisfiable. On the other hand, suppose that Γ is unsatisfiable. Then, $Mod(\Gamma) = \emptyset$. It follows that $Mod(\mathsf{KForget}(\Gamma, V)) = \emptyset$.

Result 2 directly follows from Definition 2 and the fact $Mod(\Gamma_1) = Mod(\Gamma_2)$.

Now we prove Result 3. Suppose that M is a k-model of $\mathsf{KForget}(\Gamma_1, V)$, then there exists a k-model M' of Γ_1, V such that $M \leftrightarrow_V M'$. Since $\Gamma_1 \models \Gamma_2$, M' is also a k-model of Γ_2. Hence, M is a k-model of $\mathsf{KForget}(\Gamma_2, V)$ as well.

To prove Result 4, we need to show $Mod(\mathsf{KForget}(\phi_1 \vee \phi_2, V)) = Mod(\mathsf{KForget}$ $(\phi_1, V) \vee \mathsf{KForget}(\phi_2, V))$. Suppose that M is a k-model of $\mathsf{KForget}(\phi_1 \vee \phi_2, V)$, then there exists a k-model M_0 of $\phi_1 \vee \phi_2$ such that $M_0 \leftrightarrow_V M$. Since M_0 is a k-model of $\phi_1 \vee \phi_2$, M_0 is a k-model of ϕ_1 or ϕ_2. Without loss of generality, suppose that M_0 is a k-model of ϕ_1. We have that M is a k-model of $\mathsf{KForget}(\phi_1, V)$. Thus, M is a k-model of $\mathsf{KForget}(\phi_1, V) \vee \mathsf{KForget}(\phi_2, V)$.

On the other hand, suppose that M as a k-model of $\mathsf{KForget}(\phi_1, V) \vee \mathsf{KForget}(\phi_2, V)$, then M is a k-model of $\mathsf{KForget}(\phi_1, V)$ or a k-model of $\mathsf{KForget}(\phi_2, V)$. Without loss of generality, suppose that M is a k-model of $\mathsf{KForget}(\phi_1, V)$, then there exists a k-model M_0 of ϕ such that $M_0 \leftrightarrow_V M$. M_0 is also a k-model of $\phi_1 \vee \phi_2$. Thus, M is a k-model of $\mathsf{KForget}(\phi_1 \vee \phi_2, V)$.

Finally we prove Result 5. Suppose that M is a k-model of $\mathsf{KForget}(\phi_1 \wedge \phi_2, V)$, then there exists a k-model M_0 of $\phi_1 \wedge \phi_2$ such that $M_0 \leftrightarrow_V M$. Therefore, M_0 is a k-model of ϕ_1. Thus, M is also a k-model of $\mathsf{KForget}(\phi_1, V)$. Similarly, M is a k-model of $\mathsf{KForget}(\phi_2, V)$ as well. $\Box$

We note that the converse of Result 5 in Theorem 3 does not hold generally. For instance, let ϕ be $q \equiv p$ and ψ be $q \equiv r$. Then, $\mathsf{KForget}(\phi \wedge \psi, \{p\})$ is equivalent to $q \equiv r$, while $\mathsf{KForget}(\phi, \{p\}) \wedge \mathsf{KForget}(\psi, \{p\}) \equiv \top$.

4 Knowledge forgetting and knowledge update

As we discussed in Section 1, knowledge forgetting and knowledge update represent two different perspectives of modeling an agent's knowledge change. In this section, we show that knowledge forgetting can be also used as a flexible logical notion to represent various knowledge updates. In the rest of this section, we will first compare our knowledge forgetting with Baral and Zhang's knowledge update, and then provide two different methods to define knowledge update operators via knowledge forgetting. We will also restrict our underlying language to be finite, in order to relate our knowledge update to traditional update postulates [17].

4.1 Comparison with Baral and Zhang's forgetting update

Traditional belief revision and update are based on classical propositional logic, e.g. [13,20]. Recently, Baral and Zhang studied the theory of update based on the finite propositional S5 modal logic, which they called *knowledge update* [1]. They considered knowledge forgetting is a special kind of update called *forgetting update*. In particular, let Γ be a knowledge set, and ϕ an objective formula, then in Baral and Zhang's update formulation, knowledge forgetting ϕ from Γ is achieved by performing the forgetting update $\Gamma \diamond (\neg K\phi \wedge \neg K\neg\phi)$. Intuitively, this means that after knowledge forgetting ϕ, the resulting knowledge set will not entail any knowledge about ϕ: neither knowing ϕ nor knowing $\neg\phi$.

Since Baral and Zhang's knowledge update deals with arbitrary propositional formulas while ours only considers atoms, we first restrict their forgetting update on atoms in order to make these two formulations comparable. We use notion $\Gamma \diamond_{BZ} (\neg Ka \wedge \neg K\neg a)$ to denote the result of forgetting update on atom a from Γ by using Baral and Zhang's approach. Then from Proposition 7 in [1], $\Gamma \diamond_{BZ} (\neg Ka \wedge \neg K\neg a)$

can be defined as follows.

Definition 4 *(**Baral and Zhang's forgetting update** [1]) A k-interpretation $M' = \langle W', w' \rangle \in Mod(\Gamma \diamond_{BZ} (\neg Ka \wedge \neg K\neg a))$ iff there exists a k-model $M = \langle W, w \rangle$ of Γ such that*

(1) $w' = w$;
(2) $W' = W \cup \{w^*\}$, where (a) $a \notin w^*$ if $M \models Ka$, or (b) $a \in w^*$ if $M \models K\neg a$;
(3) $W' = W$ if $M \not\models Ka$ and $M \not\models K\neg a$.

Example 2 *Suppose $\Gamma \equiv Kb \wedge (Ka \vee K\neg a)$. Let $w_0 = \{a, b\}$, $w_1 = \{b\}$, $w_2 = \{a\}$ and $w_3 = \emptyset$. Clearly, Γ has two k-models: $M_0 = \langle \{w_0\}, w_0 \rangle$ and $M_1 = \langle \{w_1\}, w_1 \rangle$. Using Definition 4, it is observed that $\Gamma \diamond_{BZ} (\neg Ka \wedge \neg K\neg a)$ has four k-models: $M'_0 = \langle \{w_0, w_1\}, w_0 \rangle$, $M'_1 = \langle \{w_0, w_3\}, w_0 \rangle$, $M'_2 = \langle \{w_0, w_1\}, w_1 \rangle$ and $M'_3 = \langle \{w_1, w_2\}, w_1 \rangle$.*

By applying knowledge forgetting definition (i.e. Definition 2), we can see that $Mod(\mathsf{KForget}(\Gamma, \{a\}))$ contains k-models M_0, M_1, M'_0, and M'_2, from which we conclude that $\mathsf{KForget}(\Gamma, \{a\}) \equiv Kb$.

From Example 2, we can see that even for this simple case, our knowledge forgetting is different from Baral and Zhang's forgetting update. In the above example, we also observe $\Gamma \diamond_{BZ} (\neg Ka \wedge \neg K\neg a) \not\models Kb$ although from our intuition knowledge forgetting a should not affect the agent's knowledge about b. In general we have the following result.

Proposition 4 *Baral and Zhang's forgetting update defined in Definition 4 does not satisfy forgetting postulates (W), (PP), (NP), and (IR).*

Proposition 4 suggests that specifying knowledge forgetting based on knowledge update, as the way proposed in [1], cannot capture the desired properties of forgetting.

4.2 Representing knowledge update via knowledge forgetting

Although Baral and Zhang's forgetting update does not satisfy our forgetting postulates (W), (PP) and (NP), their knowledge update satisfies traditional Katsuno and Mendelzon's update postulates (U1)-(U8) [1,17]. In the following, we consider how we can represent knowledge update through the notion of knowledge forgetting and whether such knowledge update satisfies these postulates. Firstly, we provide Katsuno and Mendelzon's update postulates as follows.

(U1) $\Gamma \diamond \phi \models \phi$.
(U2) If $\Gamma \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma$.
(U3) If both Γ and ϕ are satisfiable, then $\Gamma \diamond \phi$ is also satisfiable.
(U4) If $\Gamma_1 \equiv \Gamma_2$ and $\phi_1 \equiv \phi_2$, then $\Gamma_1 \diamond \phi_1 \equiv \Gamma_2 \diamond \phi_2$.

(U5) $(\Gamma \diamond \phi) \wedge \psi \models \Gamma \diamond (\phi \wedge \psi)$.
(U6) If $\Gamma \diamond \phi \models \psi$ and $\Gamma \diamond \psi \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma \diamond \psi$.
(U7) If Γ has a unique k-model, then $(\Gamma \diamond \phi) \wedge (\Gamma \diamond \psi) \models \Gamma \diamond (\phi \vee \psi)$.
(U8) $(\Gamma_1 \vee \Gamma_2) \diamond \phi \equiv (\Gamma_1 \diamond \phi) \vee (\Gamma_2 \diamond \phi)$.

A straightforward way to define knowledge update via forgetting seems as follows. Let Γ be a knowledge set and μ a satisfiable S5 formula. Γ is updated by μ, denoted as $\Gamma \diamond_{k_1} \mu$, is a formula satisfying the following condition:

$$Mod(\Gamma \diamond_{k_1} \mu) = \bigcup_{V_{min}} Mod(\mathsf{KForget}(\Gamma, V_{min}) \wedge \mu)), \tag{1}$$

where $V_{min} \subseteq Var(\Gamma)$ is a minimal subset of $Var(\Gamma)$ such that $\mathsf{KForget}(\Gamma, V_{min}) \wedge \mu)$ is consistent.

Under the definition of $\diamond_{k_1}$, the update of Γ with formula μ is achieved by knowledge forgetting all minimal subsets of atoms V_{min} from Γ while keeping $\mathsf{KForget}(\Gamma, V_{min}) \wedge \mu$ consistent.

Example 3 *Let us consider the knowledge update problem discussed in Example 2 once again.* $\Gamma \equiv Kb \wedge (Ka \vee K\neg a)$ *and* $\mu \equiv \neg Ka \wedge \neg K \neg a$. *As showed in Example 2,* $\Gamma \diamond_{BZ} \mu$ *has four k-models:* $M'_0 = \langle \{w_0, w_1\}, w_0 \rangle$, $M'_1 = \langle \{w_0, w_3\}, w_0 \rangle$, $M'_2 = \langle \{w_0, w_1\}, w_1 \rangle$ *and* $M'_3 = \langle \{w_1, w_2\}, w_1 \rangle$.

Now we consider $\Gamma \diamond_{k_1} \mu$. *Clearly,* $V_{min} = \{a\}$ *is the only minimal set of atoms that retains* $\mathsf{KForget}(\Gamma, V_{min}) \wedge \mu$ *to be consistent. So we have* $\mathsf{KForget}(\Gamma, V_{min}) \equiv Kb$ *and hence* $\Gamma \diamond_{k_1} \mu \equiv Kb \wedge \neg Ka \wedge \neg K \neg a$*, which has two k-models* $M'_1 = \langle \{\{a, b\}, \{b\}\}, \{a, b\} \rangle$ *and* $M'_2 = \langle \{\{a, b\}, \{b\}\}, \{b\} \rangle$.

Theorem 4 *Knowledge operator* $\diamond_{k_1}$ *satisfies Katsuno and Mendelzon's update postulates (U1) - (U7), but does not satisfy postulate (U8).*

Proof. It is easy to show that $\diamond_{k_1}$ satisfies (U1)-(U4). Now we prove (U5). Suppose that M is a k-model of $(\Gamma \diamond_{k_1} \phi) \wedge \psi$. Then there exists V which is minimal and M is a k-model of $\mathsf{KForget}(\Gamma, V) \wedge \phi$. Thus, M is a k-model of $\mathsf{KForget}(\Gamma, V) \wedge \phi \wedge \psi$. Therefore V is also a minimal set of atoms such that $\mathsf{KForget}(\Gamma, V) \wedge \phi \wedge \psi$ is consistent. This shows that M is also a k-model of $\Gamma \diamond_{k_1} (\phi \wedge \psi)$.

Now we prove (U6). Suppose that M is a k-model of $\Gamma \diamond_{k_1} \phi$. Then, M is also a k-model of ψ. There exists V which is minimal and M is a k-model of $\mathsf{KForget}(\Gamma, V) \wedge \phi$. Therefore M is a k-model of $\mathsf{KForget}(\Gamma, V) \wedge \psi$. This shows that $\mathsf{KForget}(\Gamma, V) \wedge \psi$ is consistent. Moreover, V is also the minimal set. Otherwise, suppose that $V_1 \subset V$ such that $\mathsf{KForget}(\Gamma, V_1) \wedge \psi$ is consistent as well. Then, $\mathsf{KForget}(\Gamma, V_1) \wedge \phi$ should also be consistent, which contradicts to the fact that V is the minimal set of atoms such that $\mathsf{KForget}(\Gamma, V) \wedge \psi$ is consistent. Hence, M is also a k-model of $\Gamma \diamond_{k_1} \psi$. Similarly, if M is a k-model of $\Gamma \diamond_{k_1} \psi$, it is a k-model of $\Gamma \diamond_{k_1} \phi$ too.

Now we prove (U7). Suppose that Γ has the unique k-model M and M_1 is the k-model of both $\Gamma \diamond_{k_1} \phi$ and $\Gamma \diamond_{k_1} \psi$. Then there exist V_1 and V_2 which are minimal such that M_1 is a k-model of both $\mathsf{KForget}(\Gamma, V_1) \wedge \phi$ and $\mathsf{KForget}(\Gamma, V_2) \wedge \psi$. Thus, $M_1 \leftrightarrow_{V_1} M$ and $M_1 \leftrightarrow_{V_2} M$. Therefore $M_1 \leftrightarrow_{V_1 \cap V_2} M$. Thus, M_1 is a k-model of $\mathsf{KForget}(\Gamma, V_1 \cap V_2)$. Therefore $V_1 = V_2$, otherwise V_1 (or V_2) is not the minimal set. M_1 is a k-model of $\mathsf{KForget}(\Gamma, V_1) \wedge (\phi \vee \psi)$ as well. Moreover, V_1 is the minimal set such that $\mathsf{KForget}(\Gamma, V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $\mathsf{KForget}(\Gamma, V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $\mathsf{KForget}(\Gamma, V_3) \wedge \phi$ or $\mathsf{KForget}(\Gamma, V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $\mathsf{KForget}(\Gamma, V_3) \wedge \phi$ is satisfiable, then V_1 is not the minimal set, a contradiction. So M_1 is also a k-model of $\mathsf{KForget}(\Gamma, V_3) \wedge (\phi \vee \psi)$.

As a counterexample of (U8), let Γ_1 and Γ_2 be $p \wedge q \wedge r$ and $p \wedge \neg r$ respectively, and ϕ be $\neg p \wedge \neg q$. We have that $(\Gamma_1 \vee \Gamma_2) \diamond_{k_1} \phi$ is $\mathsf{KForget}((p \wedge q \wedge r) \vee (p \wedge \neg r), p) \wedge (\neg p \wedge \neg q)$, which is equivalent to $\neg p \wedge \neg q \wedge \neg r$. However, $(\Gamma_1 \diamond_{k_1} \phi) \vee (\Gamma_2 \diamond_{k_1} \phi)$ is $(\mathsf{KForget}(p \wedge q \wedge r, \{p, q\}) \wedge (\neg p \wedge \neg q)) \vee (\mathsf{KForget}(p \wedge \neg r, p) \wedge (\neg p \wedge \neg q))$, which is equivalent to $\neg p \wedge \neg q$. They are not equivalent to each other. $\Box$

Theorem 4 reveals that the knowledge update specified through knowledge forgetting in such a way of (1) does not precisely capture the update semantics. This is not very surprising, because from the definition of operator $\diamond_{k_1}$, we observe that $\diamond_{k_1}$ seems not to be associated to any Γ's k-model based pre-ordering (see Theorem 3.4 in [17]).

In the following, we will propose another method of defining knowledge update via knowledge forgetting which will satisfy all postulates (U1)-(U8). For this purpose, we first specify a formula which completely characterizes a given (finite) k-interpretation. Let π be an interpretation and V a finite set of atoms, the *characteristic formula* of π on V, denoted by $C(\pi, V)$, is defined as:

$$\bigwedge_{a \in \pi, a \in V} a \wedge \bigwedge_{b \notin \pi, b \in V} \neg b.$$

It is clear that $\pi \models C(\pi, V)$.

Now consider a (finite) k-interpretation $M = \langle W, w \rangle$ and a finite set V of atoms. Then the *characteristic formula* of M on V, denoted by $C(M, V)$, is defined as:

$$C(w, V) \wedge \bigwedge_{\{w' \in W\}} \neg K \neg C(w', V) \wedge \\ \bigwedge_{\{\forall w'' \in 2^{Atom} \wedge w'' \notin W, \nexists w^* \in W s.t. w'' \leftrightarrow_{Atom \setminus V} w^*\}} K \neg C(w'', V).$$

It is not difficult to see that $M \models C(M, V)$. We can also prove that for any k-interpretation M', $M' \models C(M, V)$ iff $M' \leftrightarrow_{Atom \setminus V} M$.

Definition 5 *Let Γ be a knowledge set and μ a satisfiable formula. The knowledge update operator $\diamond_{k_2}$ is defined as follows:*

$$Mod(\Gamma \diamond_{k_2} \mu) = \bigcup_{M \in Mod(\Gamma)} \bigcup_{V_{min}} Mod(\mathsf{KForget}(C(M, Atom), V_{min}) \wedge \mu), \quad (2)$$

where $C(M, Atom)$ is the characteristic formula of M, and $V_{min} \subseteq Atom$ is a minimal subset of atoms that makes $\mathsf{KForget}(C(M, Atom), V_{min}) \wedge \mu$ consistent.

Knowledge update defined in Definition 5 can be viewed as a model based update specification, since $\Gamma \diamond_{k_2} \mu$ is achieved by minimally changing every k-model of Γ to make it consistent with μ. Note that the characteristic formula $C(M, Atom)$ is the syntactic representation of model M, and $\mathsf{KForget}(C(M, Atom), V_{min})$ guarantees the change of M is minimal with respect to μ.

By comparing (1) and (2), we can see the difference between these two knowledge update definitions: in (1) the minimal set V_{min} of forgotten atoms does not depend on any particular k-model of Γ, while in (2), each k-model of Γ has a corresponding minimal set of forgotten atoms.

Now we can define a partial ordering over the set of k-interpretations that links to knowledge operator $\diamond_{k_2}$. Let M, M_1, and M_2 be three k-interpretations. We say that M_1 is *at least as close to M as M_2* is, denoted by $M_1 \leq_M M_2$, iff for any $V_2 \subseteq Atom$ such that $M_2 \leftrightarrow_{V_2} M$, there exists a $V_1 \subseteq Atom$ such that $M_1 \leftrightarrow_{V_1} M$ and $V_1 \subseteq V_2$. We denote $M_1 <_M M_2$ iff $M_1 \leq_M M_2$ and $M_2 \not\leq_M M_1$.

Proposition 5 *Let M be a k-interpretation. Then $\leq_M$ defined in Definition 5 is a partial ordering.*

Let $\mathcal{M}$ be a collection of all k-interpretations and M a k-interpretation, we use $Min(\mathcal{M}, \leq_M)$ to denote the set of all minimal k-interpretations with respect to ordering $\leq_M$. Then we have the following theorem.

Theorem 5 *Let Γ be a knowledge set and μ a satisfiable S5 formula. Then $Mod(\Gamma \diamond_{k_2} \mu) = \bigcup_{M \in Mod(\Gamma)} Min(Mod(\mu), \leq_M)$.*

Proof. Consider a k-model $M' \in Mod(\Gamma \diamond_{k_2} \mu)$. We show that there exists some $M \in Mod(\Gamma)$ such that $M' \in Min(Mod(\mu), \leq_M)$. Definition 5, we know that there exists some $M \in Mod(\Gamma)$ such that $M' \in \bigcup_{V_{min}} Mod(\mathsf{KForget}(C(M, Atom), V_{min}) \wedge \mu)$. Further, there is a particular $V'_{min} \subseteq Atom$ such that $M' \leftrightarrow_{V'_{min}} M$ and $M' \in Mod(\mu)$. Since such V'_{min} is a minimal subset of $Atom$ satisfying these properties, it concludes that for any other k-model M'' of μ, we have $M' \leq_M M''$, that is $M' \in Min(Mod(\mu), \leq_M)$.

Now we consider a k-model $M' \in \bigcup_{M \in Mod(\Gamma)} Min(Mod(\mu), \leq_M)$. Then there exists some $M \in Mod(\Gamma)$ such that $M' \in Min(Mod(\mu), \leq_M)$. Let V_{min} be a minimal subset of atoms such that $M' \leftrightarrow_{V_{min}} M$. Then according to the definition of $\leq_M$, we know that there does not exist another k-model $M'' \in Mod(\mu)$ such that $M'' \leftrightarrow_{V''} M$ and $V'' \subset V_{min}$. This follows that $M' \in Mod(\mathsf{KForget}(C(M, Atom),$

$V_{min}) \cap Mod(\mu)$. So $M' \in Mod(\Gamma \diamond_{k_2} \mu)$. $\square$

From Theorem 5, we can show that the following result holds for $\diamond_{k_2}$.

Theorem 6 *Knowledge update operator $\diamond_{k_2}$ satisfies Katsuno and Mendelzon's update postulates (U1) - (U8).*

Example 4 *Consider a knowledge set $\Gamma \equiv K(a \wedge b \wedge c) \vee K(a \wedge \neg b \wedge \neg c)$, and a formula $\mu \equiv K\neg a \wedge K\neg b$. Now we first consider the update of Γ with μ under the operator $\diamond_{k_1}$. It is easy to see that $V_{min} = \{a\}$ is the only minimal set of atoms that makes $\mathsf{KForget}(\Gamma, V_{min}) \wedge \mu$ consistent. That is, we have $\Gamma \diamond_{k_1} \mu \equiv K\neg a \wedge K\neg b \wedge K\neg c$.*

Now we consider the same update under operator $\diamond_{k_2}$. Since Γ has two k-models: $M_1 = \langle\{\{a, b, c\}\}, \{a, b, c\}\rangle$ and $M_2 = \langle\{\{a\}\}, \{a\}\rangle$, $Mod(\mathsf{KForget}\ (C(M_1, \{a, b, c\}), \{a, b\}) \wedge (K\neg a \wedge K\neg b))$ contains a unique k-interpretation $M_1' = \langle\{\{c\}\}, \{c\}\rangle$, where $V_{min} = \{a, b\}$ is the only minimal set of atoms that makes $\mathsf{KForget}(C(M_1, \{a, b, c\}), V_{min}) \wedge (K\neg a \wedge K\neg b)$ consistent, and $Mod(\mathsf{KForget}\ (C(M_2, \{a, b, c\}), \{a\}) \wedge (K\neg a \wedge K\neg b))$ contains a unique k-interpretation $M_2' = \langle\{\emptyset\}, \emptyset\rangle$. This gives that $Mod(\Gamma \diamond_{k_2} \mu) = \{M_1', M_2'\}$. That is, $\Gamma \diamond_{k_2} \mu \equiv K\neg a \wedge K\neg b \wedge (Kc \vee K\neg c)$.

Comparing the results of $\Gamma \diamond_{k_1} \mu \equiv K\neg a \wedge K\neg b \wedge K\neg c$ and $\Gamma \diamond_{k_2} \mu \equiv K\neg a \wedge K\neg b \wedge (Kc \vee K\neg c)$, it seems that the later gives a more intuitive knowledge update solution: since $\Gamma \models Kc \vee K\neg c$, and atom c does not occurs in μ, we would expect that the resulting knowledge set still entails $Kc \vee K\neg c$. This is true for $\Gamma \diamond_{k_2} \mu$ but not for $\Gamma \diamond_{k_1} \mu$.

It is worth mentioning that knowledge update operator $\diamond_{k_2}$ is defined in a spirit of the traditional possible models approach (PMA), but under S5 model semantics. By restricting to a propositional language, $\diamond_{k_2}$ coincides with Doherty *et al.*'s propositional belief update operator [9,16] - a modified version of PMA.

5 A knowledge game with bounded memory

As indicated by Fagin *et al.* [11], there are scenarios where an agent may have to forget some facts that he knows previously. To show how such a scenario can be modeled using the theory of knowledge forgetting, in this section we study a specific knowledge game with bounded memories in a finite language. The issue of knowledge games has been extensively studied by van Ditmarsch [4], in which information contained in a game state and information change due to a game action are specifically considered. While memory is usually not an issue in knowledge games, i.e. each agent will never forget his knowledge during a game play, here we will consider that in a game, the player only has bounded memory, therefore in order to continue his play in a game, he may have to forget some of his previous knowledge.

5.1 *The game*

The knowledge game we consider here is a simple card game over a finite domain with only one player. The game is described informally as follows. There are N different cards, each with a number from 1 to N (the player knows that). Before the game starts, these N cards are facing down on the table so that the player cannot see the numbers on these cards. Then the player starts the game by picking up cards from the table, one at a time (of course the player then can see the numbers on the cards he is holding).

However, the player can at most hold M cards at the same time ($M < N$). That is, if the player already holds M cards and wants to continue the game, he has to discard one or more cards in order to pick up a new card from the table. The discarded cards then cannot be used any more. The player may pick up at most P cards ($M \leq P \leq N$) during a game. The game *terminates* if the player stops the game (e.g. the player has realised that he has won the game), or the player has exhausted all his P cards. The player *wins* the game if when the game terminates, the player holds M cards and these cards have such a property: $sum \in X$, where sum is the sum of all numbers on these M cards, and X is a set of integers [3]. Otherwise, the player loses the game.

Although this card game looks similar to those proposed by van Ditmarsch, such as Pit [5], what makes this one different from other card games (knowledge games), is that the player has a bounded memory which leads him to only have imperfect recall. That is, during a game, once the player discards a card, he will *forget* the number on that card. Then such forgetting will influence the player's current knowledge about the game. We call a game like this a *memory bounded card game*, and denote as $\mathcal{G}_X^{N,M,P}$, where parameters N, M, P and X are as described above.

5.2 *Game states, game actions and game instances*

In the following, we will formalize this card game. In particular, we consider game $\mathcal{G}_{\{6,7\}}^{5,2,3}$. That is, there are 5 cards named a, b, c, d and e, each with a number from 1 to 5, the player can only hold 2 cards at the same time, and the player can at most pick up 3 cards altogether. The player wins the game if, at the end, he holds two cards having sum of 6 or 7. Results presented in this section can be extended to general cases.

In our formalism, atoms a, b, c, d, e denote that the player holds cards a, b, c, d, e respectively, and atoms $sum_{a+b} = 3$, $sum_{a+c} = 4$, $\cdots$, $sum_{d+e} = 9$ denote the sums of cards a and b, a and c, $\cdots$, d and e respectively. Finally atoms $guess(a) = 1$, $guess(a) = 2$, $\cdots$, are used to denote the player's guess of card a's number is 1,

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[3] We assume that among the given N cards, there exists at least one collection of M different cards satisfying this property.

$2, \cdots$.

For convenience, we also introduce *meta variables* in our descriptions such as $x, y, \cdots$, and $n, n(x), n(y), n(x+y), \cdots$, where $x, y, \cdots$, are cards from $\{a, b, c, d, e\}$, $n, n(x), n(y), \cdots$, are cards' numbers from $\{1, 2, 3, 4, 5\}$, and $n(x+y), \cdots$, are sums of two cards x and y from $\{3, \cdots, 9\}$ respectively. In this way, we may write $guess(x) = n$, $\neg guess(y) = n(x)$, $sum_{x+y} = n(x+y)$ and so on to express that the player's guess on card x's number is n, the player will never guess card y's number is the same as card x's number, the sum of cards x and y's numbers is $n(x+y)$, etc..

A *game state* in a $\mathcal{G}^{5,2,3}_{\{6,7\}}$ game is a k-interpretation (W, w), where W is the set of all worlds that the player thinks possible, and w is the actual world where the player is. Since we are formalizing the situation from the player's viewpoint where the player does not know his actual world w, here we may simply refer W to the game state. In an arbitrary $\mathcal{G}^{5,2,3}_{\{6,7\}}$ game, the player will always start the same initial state: he has no card in hand, and all cards are on the table. Formally, we specify the *initial game state* W_{init} of a $\mathcal{G}^{5,2,3}_{\{6,7\}}$ game to be the set of worlds, where $w \in W_{init}$ iff $w \models \bigwedge_{x\in\{a,b,c,d,e\}}(guess(x) = 1 \oplus \cdots \oplus guess(x) = 5) \wedge \bigwedge_{x\in\{a,b,c,d,e\}} \neg x$ [4]. Note that if $w \in W_{init}$, then for any other world w' such that $w' \leftrightarrow_V w$, where $V = \{sum_{a+b} = 3, \cdots, sum_{d+e} = 9\}$, $w' \in W_{init}$.

In a $\mathcal{G}^{5,2,3}_{\{6,7\}}$ game, the player can take three kinds of *game actions*: pick up a card, discard a card, and stop the game. For a given game state W, a game state W' is a *successor game state* (or simply called successor state) of W if W' represents a game state resulting from W by taking one of the three game actions, denoted as $W' = succ(W, \alpha)$, where

$$\begin{aligned}\alpha \in \mathsf{Action} = \{&pickup(a), \cdots, pickup(e), \cdots, discard(a), \cdots,\\ &discard(e), stop\}.\end{aligned}$$

The following task is to specify $succ(W, \alpha)$ for a given W and α. Towards this aim, we first introduce a useful notion. Let y_1, y_2, y_3, and y_4 be four different cards from $\{a, b, c, d, e\}$, X and Y are two proper subsets of $\{1, 2, 3, 4, 5\}$ such that $0 \leq |X| \leq 2$ and $0 \leq |Y| \leq 1$. Then we use $\tau(y_1, y_2, y_3)|\{1, 2, 3, 4, 5\}^{-X}$ (or $\tau(y_1, y_2, y_3, y_4)|\{1, 2, 3, 4, 5\}^{-Y}$) to denote an arbitrary *guess* of the numbers on cards y_1, y_2 and y_3 (or y_1, y_2, y_3, and y_4, resp.) from $\{1, 2, 3, 4, 5\}$ but not including any numbers from X (or Y resp.). For instance, the following are two possible guesses:

$$\begin{aligned}&\tau(c, d, e)|\{1, 2, 3, 4, 5\}^{-\{2\}} = \{guess(c) = 1, guess(d) = 3, guess(e) = 4\},\\ &\tau(b, c, d, e)|\{1, 2, 3, 4, 5\}^{-\{1\}} = \{guess(b) = 3, guess(c) = 2,\\ &\qquad\qquad\qquad\qquad\qquad\qquad guess(d) = 4, guess(e) = 5\}.\end{aligned}$$

[4] Here $\oplus$ is exclusive or.

Definition 6 *(***Game actions***) Let W be a game state, $succ(W, pickup(x))$, $succ(W, discard(x))$ and $succ(W, stop)$, where $x \in \{a, b, c, d, e\}$ are defined as:*

(1) There are two cases for specifying $succ(W, pickup(x))$:
(a) if $\forall w \in W, \exists y \in w$, where $y \in \{a, b, c, d, e\} \backslash \{x\}$, then $succ(W, pickup(x))$
$= \{w^* \mid \forall w \in W$ such that $(guess(x) = n(x), sum_{x+y} = n(x+y) \in w)$,
$w^* = w \cup \{x\} \setminus \{guess(x) = n(x)\}\}$;
(b) if $\forall w \in W, \nexists y \in w$, where $y \in \{a, b, c, d, e\} \backslash \{x\}$, then $succ(W, pickup(x))$
$= \{w^* \mid \forall w \in W$ such that $(guess(x) = n(x) \in w)$,
$w^* = w \cup \{x\} \setminus \{guess(x) = n(x)\}\}$;

(2) There are two cases for specifying $succ(W, discard(x))$:
(a) if $\forall w \in W, \exists y \in w$, where $y \in \{a, b, c, d, e\} \backslash \{x\}$, then $succ(W, discard(x))$
$= \{w^* \mid w^* = \{y, \tau(y_1, y_2, y_3) | \{1, 2, 3, 4, 5\}^{-\{n(y)\}}\} \cup X$, where
$y_1, y_2, y_3 \in \{a, b, c, d, e\} \setminus \{x, y\}$, and X is any subset
of $\{sum_{a+b} = 3, \cdots, sum_{d+e} = 9\}\}$;
(b) if $\forall w \in W, \nexists y \in w$, where $y \in \{a, b, c, d, e\} \backslash \{x\}$, then $succ(W, discard(x))$
$= \{w^* \mid w^* = \{\tau(y_1, y_2, y_3, y_4) | \{1, 2, 3, 4, 5\}^{-\emptyset}\} \cup X$, where
$y_1, y_2, y_3, y_4 \in \{a, b, c, d, e\} \setminus \{x\}$ and X is any subset
of $\{sum_{a+b} = 3, \cdots, sum_{d+e} = 9\}$;

(3) $succ(W, stop) = W$.

Let us take a closer look at condition (1) in Definition 6. Consider the definition of $succ(W, pickup(a))$ first. Suppose the current game state of the player is W, in which the player already holds one card, say card b for example. Then after the player picks up card a, the player should hold both cards a and b, and thus the player also knows the sum sum_{a+b} of a and b. At this time, the player does not guess the number on card a any more, but still has to guess other cards' numbers that are not in his hand. Condition (1.a) in Definition 6 exactly captures this intuition. If the player does not hold any card at the state W, then after picking up card a, the only information change is that the player does not guess card a's number, while other information represented in W does not change, as condition (1.b) in Definition 6 shows. A similar explanation follows for condition (2). Finally, since there is no information change when the player stops the game, we have $succ(W, stop) = W$.

Definition 7 *(***Game instance***) A* game instance *of $\mathcal{G}^{5,2,3}_{\{6,7\}}$ is a finite sequence of game states associating with a sequence of game actions:*

$$\mathcal{I} = \langle W_1, W_2, \cdots, W_k \rangle | \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1},$$

where $W_1 = W_{init}$, and for each i $(1 \leq i < k)$, $W_{i+1} = succ(W_i, \alpha_i)$ for some $\alpha_i \in \mathsf{Action}$.

In Definition 7, we call W_k the *final game state* of instance $\mathcal{I}$. Note that if the player stops the game, then we always have $W_k = succ(W_{k-1}, stop) = W_{k-1}$. However, the game may also terminate if the player has exhausted his three cards. For a given S5 formula ϕ and a game instance $\mathcal{I}$, we also write $W \models_{\mathcal{I}} \phi$ if $W \models \phi$ and W is a game state in $\mathcal{I}$.

Example 5 *The following is a game instance for game $\mathcal{G}^{5,2,3}_{\{6,7\}}$:*

$$\mathcal{I}_1 = \langle W_1, W_2, W_3, W_4, W_5, W_6\rangle | pickup(a) \cdot pickup(b) \cdot discard(a) \cdot pickup(c) \cdot stop.$$

It is easy to see that $W_6 \models_{\mathcal{I}_1} Ksum_{b+c} = 5$, from which we know that the player loses the game in $\mathcal{I}_1$.

5.3 Knowledge characteristic of game $\mathcal{G}^{5,2,3}_{\{6,7\}}$

We have provided a semantic description for knowledge game $\mathcal{G}^{5,2,3}_{\{6,7\}}$ by using k-interpretations. To reason about the player's knowledge in a game instance, we would also prefer a logical account to characterize this semantic description, so that reasoning about the player's knowledge can be carried over at a logical level.

Since we deal with a finite language, from a given game state W, we always can construct a knowledge set T that completely characterizes the information represented in W. That is, for any S5 formula φ, we will have

$$W \models \varphi \text{ iff } T \models \varphi, \tag{3}$$

as showed in [4]. However, such construction will make T exponentially large - the same size of W. As an alternative, we would like to define such knowledge set T in a succinct and syntactic way. But having a succinct characteristic of a game state will no longer guarantee that (3) always holds. In the following, we will specify an alternative characteristic criterion which is weaker than (3) but still effective enough for our purpose.

A formula is called *knowledge formula* if it is of the form $K\phi$ where ϕ is objective. Under our knowledge game extent, although arbitrary S5 formulas will be used in describing game constraints and states (see the following), we can see that mainly knowledge formulas are of interests in *reasoning* about the player's knowledge in a game instance.

Definition 8 *(***Knowledge characteristic***) Let T be a knowledge set and W a k-interpretation*[5]*. T is called a* knowledge characteristic *of W if for any knowledge formula $K\phi$, $T \models K\phi$ if and only if $W \models K\phi$.*

Obviously, if T is a knowledge characteristic of W, T can be viewed as a complete characterization of W in terms of knowledge formulas. We say that a k-interpretation W is a *maximal k-model* of T, if $W \in Mod(T)$ and there does not

[5] As we mentioned earlier, the actual world in a k-interpretation is not interested to represent a game state.

exist another k-model $W' \in Mod(T)$ such that $W \subset W'$ (i.e. proper set inclusion). Then the following theorem is important for identifying a knowledge characteristic for a given k-interpretation.

Theorem 7 *Let T be a knowledge set and W a k-interpretation. Then T is a knowledge characteristic of W if W is the unique maximal k-model of T.*

Proof. Suppose W is the unique maximal k-model of T. Since $W \in Mod(T)$, then for any formula $K\phi$ where ϕ is propositional, $T \models K\phi$ implies $W \models K\phi$. Now consider that $T \not\models K\phi$. In this case, there must be some k-model of T, say $W' \in Mod(T)$, such that $W' \not\models K\phi$. Since W is the unique maximal k-model of T, we have $W' \subseteq W$. This follows that $W \not\models K\phi$. $\Box$

Definition 9 *(***Knowledge characteristic of instance** $\mathcal{I}$*) Let $\mathcal{I} = \langle W_1, \cdots, W_k\rangle \mid \alpha_1 \cdots \alpha_{k-1}$ be an instance of game $\mathcal{G}^{5,2,3}_{\{6,7\}}$, and $\mathcal{T} = \langle T_1, \cdots, T_k\rangle$ a sequence of knowledge sets. We say that $\mathcal{T}$ is a* knowledge characteristic *of instance $\mathcal{I}$ if for each i ($1 \leq i \leq k$), T_i is a knowledge characteristic of game state W_i.*

In the following, we will show that we can use our knowledge forgetting effectively define such $\mathcal{T}$ mentioned in Definition 9. We first specify a knowledge set T_c consisting of *game constraints* as follows:

$$\begin{aligned}
&\mathsf{holdCard} \equiv \textstyle\bigwedge_{x\in\{a,b,c,d,e\}}(x \rightarrow (Kx \wedge \bigwedge_{n\in\{1,2,3,4,5\}} K\neg guess(x) = n \wedge \\
&\qquad\qquad \textstyle\bigwedge_{y\in\{a,b,c,d,e\}, y\neq x} K\neg guess(y) = n(x))), \\
&\mathsf{knowSum} \equiv \textstyle\bigwedge_{x,y\in\{a,b,c,d,e\}, x\neq y}(Kx \wedge Ky \rightarrow Ksum_{x+y} = n(x+y)), \\
&\mathsf{know_not_holdCard} \equiv \textstyle\bigwedge_{x\in\{a,b,c,d,e\}}(\neg x \rightarrow K\neg x), \\
&\mathsf{not_knowSum} \equiv \textstyle\bigwedge_{x\in\{a,b,c,d,e\}}(\neg x \rightarrow \\
&\qquad\qquad \textstyle\bigwedge_{y\in\{a,b,c,d,e\}, y\neq x} \neg Ksum_{x+y} = n(x+y)), \\
&\mathsf{guessCard} \equiv \textstyle\bigwedge_{x,y\in\{a,b,c,d,e\}, x\neq y, n\in\{1,2,3,4,5\}} K\neg(guess(x) = n \wedge guess(y) = n).
\end{aligned}$$

T_c represents basic game constraints that every game state in any game instance should satisfy. The intuitive meaning of these formulas is quite obvious. For instance, holdCard states that if the player holds a card, then he knows that he is holding that card. In this case the player will not need to guess the card's number, and also the player should then not guess this card's number for other cards. knowSum says that if the player has held two cards, then he knows the sum of these two cards' numbers. On the other hand, not_knowSum indicates that if the player does not hold a card, then the player does not know the sum of this card's number with any other card's numbers. Finally, guessCard simply says that the player will never guess two cards with the same number.

Now we specify T_0 to be a knowledge set consisting of the following formulas:

$\bigwedge_{x\in\{a,b,c,d,e\}} \neg x$, and

$\bigwedge_{x\in\{a,b,c,d,e\}} K(\bigoplus_{n\in\{1,2,3,4,5\}} guess(x) = n)$,

and let $T_{init} = T_0 \wedge T_c$ be the *initial* knowledge set that represents the initial game state W_{init} for any game instances.

Proposition 6 *T_{init} is a knowledge characteristic of W_{init}.*

Proof. According to Theorem 7, it is sufficient to prove that W_{init} is the unique maximal k-model of T_{init}. From the fact that $T_{init} = T_0 \wedge T_c$, it is easy to see that for any k-model W of T_{init} and each world $w \in W$, w must contain one of the following variables from $\{guess(a) = n_1, guess(b) = n_2, guess(c) = n_3, guess(d) = n_4, guess(e) = n_5\}$, where $n_i \in \{1, 2, 3, 4, 5\}$; w does not contain any variables from $\{a, b, c, d, e\}$; and w may or may not contain variables from $\{sum_{a+b} = 3, \cdots, sum_{d+e} = 9\}$. This means that each world $w \in W$ is also in W_{init}. So W_{init} is the unique maximal k-model of T_{init}. $\Box$

Suppose that a given T is a knowledge characteristic of some game state W, and $W' = succ(W, \alpha)$ is a successor state of W by taking action α. Then by using knowledge forgetting, we can derive a new knowledge set T' from T which is also a knowledge characteristic of W'. To begin with, we first introduce the following notions:

$$\Delta(x) = \{x\} \cup \bigcup_{n\in\{1,2,3,4,5\}}\{guess(x) = n\} \cup \bigcup_{y\in\{a,b,c,d,e\},y\neq x}\{sum_{x+y} = n(x + y)\}, \text{ and}$$
$$\Theta(x) = \Delta(x) \cup \bigcup_{y\in\{a,b,c,d,e\}\setminus\{x\},n\in\{1,2,3,4,5\}}\{guess(y) = n\}.$$

Intuitively, $\Delta(x)$ is the set of atoms that contains card x and other atoms that are directly associated to x, while $\Theta(x)$ contains some extra atoms whose truth values may be affected by discarding card x. Now we further specify a formula:

$$\mathsf{persist}(T, \{x\}) = \bigwedge_{y\in\{a,b,c,d,e\}\setminus\{x\}}((T \to \bigoplus_{n\in\{1,2,3,4,5\}} guess(y) = n) \to \bigoplus_{n\in\{1,2,3,4,5\}} guess(y) = n).$$

The intuitive meaning of $\mathsf{persist}(T, \{x\})$ is that if at some game state (that is characterized by T) the player has a guess for any card from $\{a, b, c, d, e\} \setminus \{x\}$, then the player should also have a guess for this card after discarding card x.

Definition 10 *(***Action-derived knowledge set***) Let T be a knowledge set and α an action. T' is called an* action-derived knowledge set *by applying α to T, if and only if T' is defined as follows:*

(1) If $\alpha = pickup(x)$, where $x \in \{a, b, c, d, e\}$, then
$T' = \mathsf{KForget}(T, \Delta(x)) \wedge x \wedge T_c$*;*

(2) If $\alpha = discard(x)$, where $x \in \{a, b, c, d, e\}$, then
$T' = \mathsf{KForget}(T, \Theta(x)) \wedge \neg x \bigwedge_{n\in\{1,2,3,4,5\}} K(\neg guess(x) = n) \wedge$

$K\mathsf{persist}(T, \{x\}) \wedge T_c$;

(3) If $\alpha = stop$*, then* $T' = T$.

Now we have the following result showing that the action-derived knowledge set defined in Definition 10 forms a knowledge characteristic for any game instance of $\mathcal{G}^{5,3,2}_{\{6,7\}}$.

Theorem 8 *Let* $\mathcal{I} = \langle W_1, \cdots, W_k\rangle | \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1}$ *be an instance of game* $\mathcal{G}^{5,2,3}_{\{6,7\}}$*, and* $\mathcal{T} = \langle T_1, \cdots, T_k\rangle$ *a sequence of knowledge sets, where* $T_1 = T_{init}$*, and each* T_{i+1} *is an action-derived knowledge set by applying action* α_i *to* T_i *(*$1 \leq i < k$*) as defined in Definition 10. Then* $\mathcal{T}$ *is a knowledge characteristic of* $\mathcal{I}$*.*

Proof. We prove by induction that for each i, W_i is the unique maximal k-model of T_i. From the proof of Proposition 6, we know that W_{init} is the unique maximal k-model of T_{init}. Assume for all $i < k$, W_i is the unique maximal k-model of T_i. We need to show that W_k is also a unique maximal k-model of T_k. There are three cases we should consider: (1) $W_k = succ(W_{k-1}, pickup(x))$, (2) $W_k = succ(W_{k-1}, discard(x))$, and (3) $W_k = succ(W_{k-1}, stop)$.

Proof for Case (3) is trivial. For Case (1), there are four subcases: (1.1) the player already holds a card y in W_{k-1}, and the player has not discarded any card before; (1.2) the player already holds a card y in W_{k-1}, but also the player has previously discarded a card y'; (1.3) before picking up card x, the player has no card in hand, and the player has not discarded any card yet; and (1.4) before picking up card x, the player has no card in hand, and the player has previously discarded a card y. For Case (2), there are two subcases: (2.1) the player has two cards x and y in hand before discarding card x; and (2.2) the player only has card x in hand before discarding x. Proofs for all these subcases are quite tedious but straightforward following the definitions and propositions presented in this section. So we omit those details. □

6 Concluding remarks

In this paper, we examined the notion of knowledge forgetting under S5 modal logic. We provided a complete characterization on knowledge forgetting through four postulates, and investigated its useful applications in knowledge updates and knowledge games.

Many related issues remain for further study. In this paper, we only addressed the problem of knowledge forgetting in modal logic S5. In a multi-agent system, it is more common that an agent not only needs to forget his own knowledge due to a memory limit, but also has to forget other agents' knowledge for various reasons. So generalizing our knowledge forgetting to the multi-agent modal logic S5 (and other multi-agent modal logics) will be a challenge. One particular concern we should take into account in this development is common knowledge which does not occur in single agent modal logic.

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