Knowledge updates: Semantics and complexity issues ⋆

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Abstract

We consider the problem of updating of an agent's knowledge. We propose a formal method of knowledge update on the basis of the semantics of modal logic S5. In our method, an update is specified according to the minimal change on both the agent's actual world and knowledge. We discuss general minimal change properties of knowledge update and show that our knowledge update operator satisfies all the update postulates of Katsuno and Mendelzon. We characterize several specific forms of knowledge update which have important applications in reasoning about change of agents' knowledge. We also examine the persistence property of knowledge and ignorance associated with knowledge update.

We then investigate the computational complexity of model checking for knowledge update. We first show that in general the model checking for knowledge update is Σ_2^P-complete. We then identify a subclass of knowledge update problems that has polynomial time complexity for model checking. We point out that some important knowledge update problems belong to this subclass. We further address another interesting subclass of knowledge update problems for which the complexity of model checking is NP-complete.

Keywords: belief revision and update, complexity, knowledge representation, knowledge update, Kripke structures, S5 modal logic

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1 Introduction

1.1 Motivation

The well-studied issues of belief update and belief revision [16] are concerned with the update and revision aspects of an agent's belief with respect to new beliefs. The notion of belief update has been used, and often serves as a guideline [15,30], in reasoning about the effect of (*world altering*) actions on the state of the world. Thus if ϕ represents the agent's belief about the world and the agent performs an action that is supposed to make ψ true in the resulting world, then the agent's belief about the resulting world can be described by $\phi \diamond \psi$, where $\diamond$ is the update operator of choice.

Now let us consider reasoning about sensing actions [26,28], which in their pure form, when executed, do not change the world, but change the agent's knowledge about the world. Let $sense_f$ be a sensing action whose effect is that after it is executed the agent knows whether f is true or not. This can be expressed as $Kf \vee K\neg f$, where K is the modal operator $Knows$. The current theory of belief updates does not tell us how to do updates with respect to such gain in knowledge due to a sensing action. In this regard note that we can not just have $f \vee \neg f$ and use the the notion of belief update, as $f \vee \neg f$ is a tautology.

A theory of knowledge update as targeted here will allow a reasoner to verify the correctness of a plan with both physical and sensing actions. The reasoner, different from the agent who will be executing the plan including sensing as prescribed by the plan, will be able to verify if the updating of a formula representing the initial state of the world (both physically and in terms of what the agent knows about it) by the effect of the actions in the plan will lead to a desired state of the world.

In the recent past there has also been some research in dynamic epistemic logics, e.g. [3,4,9–12,29,34,35] where the changes in *information states (of agents)* is the main focus but changes in the real world are usually not modeled.

The main goal of this paper is to define a notion of *knowledge update*, analogous to belief update, where the original theory (say α) and the new theory (say β) are in a language that can express knowledge, and *changes are allowed in both the real world and the agent's knowledge about the world.* Such a notion would not only serve as a guideline to reason about pure and mixed sensing actions in the presence of constraints, but also allow us to reason about actions

⋆ This paper is an expanded and revised version of the authors' papers in IJCAI-2001 [1] and KR-2002 [2].

corresponding to *forgetting* and *ignorance*.

In this paper we investigate the model theoretic semantics and the associated reasoning and complexity properties of such knowledge update. This not only provides the theoretical foundation to enhance the current robotic planning paradigm, as has been observed by other researchers, e.g. [20,26–28], but can also be viewed as another approach for modeling knowledge dynamics.

1.2 Our work vs the research in dynamic epistemic logic – a brief overview

Since most of the research in dynamic epistemic logics has been reported outside Artificial Intelligence avenues, in this subsection we give a brief overview of that direction of research and how it compares to our goals.

While research on reasoning about knowledge has made significant progress in the last decade, e.g. [6,13,17,21,25], the problem of modeling the dynamics of knowledge has only received attention in recent years from different perspectives. One of the major motivations of studying knowledge dynamics is for the purpose of modeling the dynamics of distributed systems. In this regard Fagin et al. [6] studied the relationship between knowledge and time from an axiomatization viewpoint where change in knowledge is caused by executing the distributed system's actions. Following this work, van der Meyden [33] also studied the computational aspect of knowledge modeling in distributed environments where the issue of knowledge update was discussed. Although van der Meyden showed that his knowledge update presented a generalization of certain aspects of standard knowledge base update, he only used it for the purpose of efficiently implementing model checking and did not explore knowledge update from a more semantical perspective.

Fagin et al's work on knowledge has recently motivated a stream of interesting investigations on dynamic epistemic logics, e.g. [3,4,9–12,29,34,35]. Changes in information states (of agents), represented by a collection of possible worlds, have been widely studied in these logics. Most of these works differ from our approach in that they usually do not model changes in the real world. The following quote from page 4 of [10] gives a feel of research in these studies:

> *In this section we will define operations on possibilities that correspond to changes in the information states of the agents. The kind of information change we want to model is that of agents getting new information and learning that the information state of some other agent has changed in a certain way. I will introduce 'programs' in the object language that describe such changes. Changes in the 'real world' will not be modeled, and I will ignore other operations of information change such as belief contraction or 'belief revision'.*

From our discussions above, we can see that knowledge dynamic modeling has been studied by many researchers in recent years. This paper can be viewed as a further study on knowledge dynamics but from a different perspective. Our focus is on the model theoretic semantics of knowledge update. This has direct applications in the field of reasoning about agents' knowledge related actions, particularly in the construction and verification of plans with sensing actions. We also explore the computational properties of various knowledge update forms so as to provide guidelines for future implementations.

Most of the previous work on knowledge dynamics and dynamic epistemic logics focus on the development of formal axiomatic systems that are able to deal with the dynamics of epistemic states and actions. Although these logics have significant applications in various multi agent environments, they seem not quite applicable for our purpose mentioned earlier. Also, the computational properties of these logics remain unexplored.

1.3 Summary of contributions of this paper

The main contributions of this paper can be summarized as follows.

(1) We define a model theoretic semantics for knowledge update based on the single agent S5 modal logic. This knowledge update semantics presents a generalization of traditional model based belief update by allowing for modalities in the base language. That is, in our framework changes on both the actual world and the knowledge state of the agent are allowable. Our underlying knowledge update operator can be characterized by an explicit minimal change principle and satisfies Katsuno and Mendelzon's classical belief update postulates [16].

(2) We characterize various forms of knowledge update such as gaining knowledge update, ignorance update, sensing update and forgetting update. Each of these update forms has its specific meaning in reasoning about agent's knowledge related actions. Furthermore, we also investigate the persistence of knowledge and ignorance during a knowledge update. Our results provide restricted monotonicity properties that may be used to simplify the underlying inference problem in knowledge update.

(3) We investigate the computational complexity of model checking for knowledge update. We show that in general the model checking problem for knowledge update is Σ_2^P-complete, which places the problem in the same layer of the polynomial hierarchy as the traditional model based belief update (e.g. PMA) [19]. We then identify a subclass of knowledge update problems for which model checking can be achieved in polynomial time. We observe that some important knowledge update problems belong to this subclass. We further address another interesting middle

class of knowledge update problem for which the complexity for model checking is NP-complete.

1.4 Structure of the reminder of the paper

The structure of the rest of the paper is as follows. In Section 2 we start with describing the particular modal logic that we plan to use in expressing knowledge, and describe the notion of k-models analogous to 'models' in classical logic. We define closeness between k-models and use it to define a particular notion of knowledge update. In Section 3 we discuss minimal change properties of knowledge update. An interesting result shows that our knowledge update operator satisfies all of the Katsuno and Mendelzon's update postulates [16]. In Section 4 we present alternative characterizations of four particular knowledge updates – *gaining knowledge, ignorance, sensing, and forgetting*, and show their equivalence to our original notion of knowledge update. Some of these alternative characterizations are based on the formulation of reasoning about sensing actions, and thus our equivalence results can serve as suitable justifications of the intuitiveness of our definition of knowledge update. In Section 5 we explore sufficiency conditions that guarantee persistence of knowledge (or ignorance) during a knowledge update. From Section 6 we start to investigate model checking complexity for knowledge update. In Section 6 we first give general background on computational complexity. In Section 7 we study the model checking complexity for the general case of knowledge update. In Section 8 we define a subclass of knowledge update problems whose model checking can be achieved in polynomial time. In Section 9 we further address an interesting intractable subclass of knowledge update problems whose model checking is lower than the general case. Finally, in Section 10 we conclude this paper with some remarks. We present proofs of all major results in an appendix.

2 Closeness between k-models and knowledge update

In this section, we describe formal definitions for knowledge update. Our formalization will be based on the semantics of the propositional modal logic S5 with a single agent. In general, under Kripke semantics, a *Kripke structure* is a triple (W, R, π), where W is a set of possible worlds, R is an equivalence relation on W, and π is a truth assignment function that assigns a propositional valuation to each world in W. Given a Kripke structure $S = (W, R, \pi)$, a *Kripke interpretation* is a pair (S, w), where $w \in W$ is referred to as the *actual world* of (S, w). To characterize S5 formulas (which we henceforth refer to simply as 'formulas') we follow [6] in defining an entailment relation $\models$

between Kripke interpretations and formulas.

In the case of a single agent, however, we restrict ourselves to those S5 structures in which the relation R is universal, i.e. each world is accessible from every world, and worlds are identified with the set of atoms true at the worlds (see page 28 in [23]). To simplify a comparison between two worlds (e.g. Definition 2), we view an atom p to be in a world w (denoted by $p \in w$) iff p is mapped to *true* in the world w (denoted by $w \models p$). Therefore, in our context a Kripke structure (W, R, π) is uniquely characterized by W and we define a k*-model* as a pair $M = (W, w)$, where w indicates the actual world of the agent and W presents all possible worlds that the agent may access. Note that since we assume R to be universal, w is in W for any k-model $M = (W, w)$.

In the rest of this paper we assume our language to have a finite number of propositions. Thus we will be dealing with a finite propositional S5 modal logic. Although, this may seem restrictive, we make this assumption to keep our focus on the main issue of the paper – investigation of knowledge updates from a viewpoint of reasoning about an agent's knowledge related actions and the associated complexity problems. Our work can be viewed as an extension of traditional propositional belief update where usually a finite language is also employed, for example in [16].

We use $a, b, c, \cdots, p, q, \cdots$ to denote propositional atoms; $\phi, \psi, \upsilon, \cdots$ to denote propositional formulas without including modalities (we also call them *objective* formulas); and $\alpha, \beta, \gamma, \mu, \cdots$ to denote formulas that may contain modal operator K. For convenience, we use $T \equiv \alpha_1 \wedge \cdots \wedge \alpha_k$ to represent a finite set of formulas $\{\alpha_1, \cdots, \alpha_k\}$ and call T a *knowledge set*.

Definition 1 *Let $\mathcal{P}$ be the set of all atomic propositions in the language. The entailment relation $\models$ under S5 semantics is defined as follows:*

(1) $(W, w) \models p$ *iff* p *is an atomic proposition (i.e.* $p \in \mathcal{P}$*) and* $w \models p$*;*
(2) $(W, w) \models \alpha \wedge \beta$ *iff* $(W, w) \models \alpha$ *and* $(W, w) \models \beta$*;*
(3) $(W, w) \models \neg\alpha$ *iff it is not the case that* $(W, w) \models \alpha$*;*
(4) $(W, w) \models K\alpha$ *iff* $(W, w') \models \alpha$ *for all* $w' \in W$*.*

We use notations and terminologies similar to the ones used in propositional logic. Following is a list of our definitions and terminologies:

- Given a formula T, (W, w) is called a k*-model of* T if $(W, w) \models T$. (Our notion of k-models is analogous to 'models' in propositional logic.)
- We use $Mod(T)$ to denote the set of all k-models of T.
  For an objective formula ϕ, $Mod(\phi)$ simply denotes the set of worlds w where $w \models \phi$. In this case, w is also called a *model* of ϕ.
- For a formula α, we say that T *entails* α, denoted as $T \models \alpha$, iff for every k-model (W, w) of T, $(W, w) \models \alpha$.

Note that in the rest of the paper, we may also use M to denote a k-model, i.e. $M = (W, w)$, and in this case we simply write $M \models \alpha$ if α is true in (W, w).

- We say a formula is *satisfiable* if it has a k-model. (This is analogous to the 'satisfiability' of propositional theories.)
- We say two formulas T and α are *equivalent*, denoted by $T \equiv \alpha$, iff $T \models \alpha$ and $\alpha \models T$.

The basic problem of knowledge update that we would like to investigate is formally described as follows: given a *k-model* $M = (W, w)$, that is usually viewed as a *state* of an agent, and a formula μ - the agent's new knowledge that may contain modal operator K, how do we update M to another k-model $M' = (W', w')$ such that $M' \models \mu$ and M' is *minimally different* from M with respect to some criterion? To define such a minimal difference (or more often called *minimal change principle*) on k-model update, we first study a concept of *closeness* between two k-models with respect to a given k-model.

A widely used definition of closeness [37] between simple worlds is based on the notion of symmetric difference. According to this definition a world w_1 is as close to the world w as w_2 is (denoted by $w_1 \leq_w w_2$) if $(w_1 \setminus w \cup w \setminus w_1) \subseteq (w_2 \setminus w \cup w \setminus w_2)$. When defining closeness of k-models we give first preference to the comparison between the actual worlds. Hence, if two k-models $M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_2)$ have different actual worlds then we define their closeness with respect to a reference k-model $M = (W, w)$ by simply comparing the symmetric difference between w_1 and w, and w_2 and w. When $w_1 = w_2$ we need additional comparisons. A straightforward approach would be to compare the knowledge encoded in each of the k-models. For that we have the following notation.

- For a k-model $M = (W, w)$, by KM we denote the set $\{\phi \mid \phi$ is an objective formula and for all $w' \in W$ we have that $w' \models \phi\}$ (here note that w is also in W).

A simple comparison between the knowledge encoded in M_1 and M_2 with respect to M can be done by comparing the symmetric difference between KM_1 and KM and KM_2 and KM. We use this comparison but in addition consider two special cases when the symmetric differences may be incomparable but yet there is reason to consider one k-model to be closer (to M) than the other. These two special cases are when M_1 only loses knowledge with respect to M, and when M_1 only gains knowledge with respect to M.

Consider the case when M_1 only loses knowledge with respect to M. In that case if M_2 both loses and gains knowledge with respect to M then we consider

M_1 to be closer to M than M_2 is to M. Also, if M_2 (like M_1) only loses knowledge with respect to M, but loses more than M_1 does then we consider M_1 to be closer to M than M_2 is to M. This is illustrated in Figure 1.

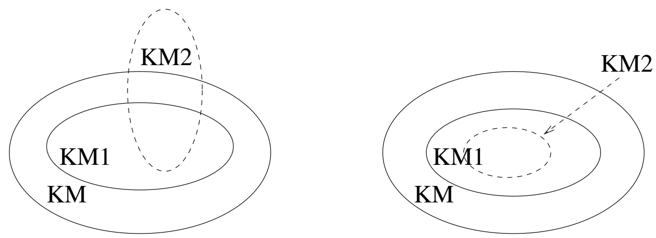

Fig. 1. $M_1 \leq_M M_2$ under the condition $w_1 = w_2$ and $W \subset W_1$.

Similarly, consider the case when M_1 only gains knowledge with respect to M. In that case if M_2 both loses and gains knowledge with respect to M then we consider M_1 to be more closer to M than M_2. Also, if M_2 (like M_1) only gains knowledge with respect to M, but gains more than M_1 does then we consider M_1 to be closer to M than M_2. This is illustrated in the following figure.

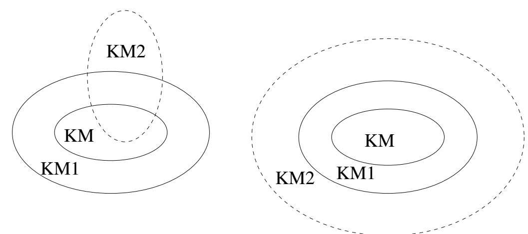

Fig. 2. $M_1 \leq_M M_2$ under the condition $w_1 = w_2$ and $W_1 \subset W$.

Note that classifying knowledge change as two special cases of only increasing (gaining) knowledge and only losing (decreasing) knowledge respectively is important in our formalization. As we will show later, several interesting knowledge update forms belong to these two types of updates. Also, the computational complexity of these two types of updates, to be discussed in Sections 8 and 9, are different. We now formally define the closeness between k-models.

Definition 2 *(***Closeness between** k**-models***) Let* $M = (W, w)$, $M_1 = (W_1, w_1)$ *and* $M_2 = (W_2, w_2)$ *be three* k*-models. We say* M_1 *is* closer or as close to M as M_2, *denoted as* $M_1 \leq_M M_2$, *if:*

(1) $(w_1 \setminus w \cup w \setminus w_1) \subset (w_2 \setminus w \cup w \setminus w_2)$*; or*
(2) $w_1 = w_2$ *and one of the following conditions holds:*
  (i) $W_1 = W_2$*;*
  (ii) $W_1 \neq W_2$ *and if* $W \subset W_1$*, then (a) there exist some* ϕ *and* ψ *such that* $M \models K\phi$ *and* $M_2 \not\models K\phi$ *and* $M \not\models K\psi$ *and* $M_2 \models K\psi$*, or (b) for any* ϕ *if* $M \models K\phi$ *and* $M_1 \not\models K\phi$*, then* $M_2 \not\models K\phi$*;*
  (iii) $W_1 \neq W_2$ *and if* $W_1 \subset W$*, then condition (a) above is satisfied, or (c) for any* ϕ *if* $M \not\models K\phi$ *and* $M_1 \models K\phi$*, then* $M_2 \models K\phi$*;*

(iv) $W_1 \neq W_2$ *and if* $W \not\subset W_1$ *and* $W_1 \not\subset W$*, then conditions (b) and (c) above are satisfied;*

(v) $W_1 \neq W_2$ *and* $W_1 = W$.

We denote $M_1 <_M M_2$ *if* $M_1 \leq_M M_2$ *and* $M_2 \not\leq_M M_1$.

Note that Fig 1 and Fig 2 given earlier illustrate the conditions (2) (ii) and (2) (iii) of the above definition respectively. The following figure illustrates the condition (2) (iv) of the above definition. The meaning of condition (2) (v), on the other hand, is quite obvious.

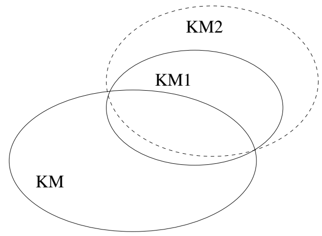

Fig. 3. $M_1 \leq_M M_2$ under the condition $w_1 = w_2$, $W \not\subset W_1$ and $W_1 \not\subset W$.

One may argue that Definition 2 above is too strong in terms of knowledge comparison, particularly the cases in condition (2)(ii) and (2)(iii) as illustrated by the first diagrams of Figure 1 and Figure 2. For instance, let $W = \{w_1, w_2, \cdots, w_{99}, w_{100}\}$, $W_1 = \{w_1\}$ and $W_2 = \{w_1, w_2, \cdots, w_{99}, w_{101}\}$, and $M = (W, w_1), M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_1)$. According to Definition 2, we have $M_1 <_M M_2$. However, it is observed that W_1 only has one world w_1 in common with W while W_2 has 99 worlds in common with W, yet it is viewed that M_1 is closer to M than M_2. Although such argument seems plausible in some sense, our motivation here is based on the agent's knowledge instead of just counting the number of common worlds between two k-models. Let us take a closer look at condition (2) (iii) which defines $M_1 <_M M_2$ in our example here. This condition actually says that (a) M_1 has all the knowledge that M has (i.e. $W_1 \subset W$); (b) there exists some knowledge that M has but M_2 does not, *and* M does not have but M_2 has; and (c) for any knowledge that M_1 has but M does not, M_2 also has. From statements (a), (b) and (c), it seems reasonable to us to conclude that M and M_1 are closer than M and M_2 do. Another way to justify our formulation is to notice that M_1 can be obtained from M by just performing a pure sensing action, while to obtain M_2 from M one has to perform a more complicated action. Similarly, M can be obtained from M_1 by a forgetting action while to obtain M from M_2 one has to perform a more complicated action. (We discuss different kinds of actions and the corresponding knowledge update in greater detail in Section 4.)

On the other hand, in Definition 2, we give higher priority to difference between actual worlds to difference between the knowledge about the worlds. This is a design decision that we made. The possibilities were to treat them equally, or treat one more important than the other. We consider changes in the real world to be harder (needs a physical action) than changes only in the knowledge. Consider three models M_1, M_2 and M. Suppose M_1 and M have the same physical part but different knowledge part; and M_2 and M have different physical parts but the same knowledge part. In this case our definition will say that M_1 is closer to M than M_2. Again, the intuition is that it is harder to change the real world through physical actions (say to break an object) than to change the knowledge (to lie and say that the object was broken). Moreover, by giving a higher priority of the real world change, our revised definition of knowledge update is consistent with updates involving only actual worlds as well as updates involving only epistemic states.

Now using the notion of closeness between k-models we define k-model update. Our definition is similar to the definition of belief updates, which is defined using closeness between worlds.

Definition 3 *(**k-model Update**) Let $M = (W, w)$ be a k-model and μ a formula. A k-model $M' = (W', w')$ is called a* possible resulting k-model *after updating M with μ if and only if the following conditions hold:*

(1) $M' \models \mu$;
(2) there does not exist another k-model $M'' = (W'', w'')$ such that $M'' \models \mu$ and $M'' <_M M'$.

We denote the set of all possible resulting k-models after updating M with μ as $Res(M, \mu)$.

Example 1 *Let $T \equiv Kc \wedge \neg Ka \wedge \neg Kb \wedge K(a \vee b)$ and $\mu \equiv K\neg c$. We denote*

$$w_0 = \{a, b, c\},\ w_1 = \{a, c\},\ w_2 = \{b, c\},$$
$$w_3 = \{c\},\ w_4 = \{a, b\},\ w_5 = \{a\},$$
$$w_6 = \{b\},\ w_7 = \emptyset.$$

Clearly, $M_0 = (\{w_0, w_1, w_2\}, w_0)$ is a k-model of T. Consider the update of M_0 with μ. Let $M_1 = (\{w_4, w_5, w_6\}, w_4)$. Now we show that M_1 is a possible resulting k-model after updating M_0 with μ.

Since $(w_0 \setminus w_4 \cup w_4 \setminus w_0) = \{c\}$, we first consider any possible k-model $M' = (W', w')$ such that $(w_0 \setminus w' \cup w' \setminus w_0) \subset \{c\}$. Clearly, the only possible w' would be w_0 itself. Let $M' = (W', w_0)$, where W' is a subset of $\{w_0, \cdots, w_7\}$. However, since $c \in w_0$, there does not exist any W' such that $M' \models K\neg c$. Therefore, from Definition 2, only condition 2 can be used to find a possible M' such that $M' <_M M_1$. So we assume $M' = (W', w_4)$. On the other hand, from M_0 and M_1, it is easy to see that $KM_0 = \{c, a \vee b\}$ and $KM_1 = \{\neg c, a \vee b\}$[2].

[2] For simplicity, here we only consider the *prime* formulas ϕ in KM in the sense that if $\phi \in KM$, then there is not another ψ such that $\models \psi \supset \phi$ and $\psi \in KM$.

Then we have $KM_0 \setminus KM_1 = \{c\}$ and $KM_1 \setminus KM_0 = \{\neg c\}$. Ignoring the detailed verifications, we can show that there does not exist such $M' = (W', w_4)$ satisfying $KM_0 \setminus KM' = KM' \setminus KM_0 = \emptyset$.

Example 2 *Let $T = a \wedge b \wedge c$, and $\mu = a \wedge b \wedge \neg c$. As in the previous example, let us assume a, b and c are the only propositions in our world. In that case we have eight possible worlds; $w_0, \ldots, w_7$, as given in the previous example.*

Now let us compute the various k-models of T. They will be of the form (W, w_0), where W is any subset of $\{w_0, \ldots, w_7\}$ containing w_0. There are $2^7 = 128$ such Ws and hence T has 128 k-models. Let M be one of these k-models, say $M = (\{w_0, \ldots, w_7\}, w_0)$. Let us now compute $Res(M, \mu)$. $Res(M, \mu)$ consists of the unique k-model $M' = (\{w_0, \ldots, w_7\}, w_4)$; as w_4 is the closest physical world to w_0 that satisfies μ, and among all other k-models of the form (W', w_4), M' is the closest to M. Note that this update does not result in any change in knowledge. Indeed, since this update does not involve K operator, the knowledge update is then reduced to the classical PMA update [37].

Using the notion of k-model update we now define the updating of a formula T by another formula μ as the union of updating every k-model of T with μ. This is similar to the way belief update is defined in the literature.

Definition 4 *(***Knowledge update***) Let T and μ be two formulas. The* update of T with μ, *denoted as $T \diamond \mu$, is defined by $Mod(T \diamond \mu) = \bigcup_{M \in Mod(T)} Res(M, \mu)$.*

Example 3 *Let $T_1 = a$, and $T_2 = Ka$. Let $w_0 = \{a\}$, $w_1 = \emptyset$.*

Let $W_0 = \emptyset$, $W_1 = \{w_0\}$, $W_2 = \{w_1\}$, and $W_3 = \{w_0, w_1\}$.

The k-models of T_1 are (W_1, w_0) and (W_3, w_0). (W_0, w_0) and (W_2, w_0) are not k-models of T_1 as neither W_0, nor W_2 contains w_0.

The only k-model of T_2 is (W_1, w_0).

Thus the only k-model of $T_1 \diamond T_2$ is (W_1, w_0).

Clearly, Definition 4 is a generalized form of Winslett's PMA update [37]. It should be noted that we would not be able to define knowledge update in such a way as in Definition 4 if we allow the underlying language to be infinite, because this would require that the set of k-models $\bigcup_{M \in Mod(T)} Res(M, \mu)$ be finitely axiomatized and this is usually not possible for infinite models. On the other hand, we may think that the update operator $\diamond$ as a function that takes formulas T and μ as parameters and *nondeterministically* returns a formula whose models are characterized by the set $\bigcup_{M \in Mod(T)} Res(M, \mu)$. In practice, it is not feasible to obtain such a specific formula as there may be infinite number of formulas whose models are represented by $\bigcup_{M \in Mod(T)} Res(M, \mu)$. Nevertheless, since our interest here is to capture the semantics of knowledge update, we actually do not need to know this particular formula. Instead, our operations on knowledge will focus on the models, similar to the approach taken in defining belief updates, e.g. [37].

3 Minimal change of knowledge update

In this section, we investigate minimal change properties of knowledge update. Specifically, we examine the relationship between knowledge update and the classical Katsuno and Mendelzon's update postulates [16]. We start with some useful results about knowledge update.

Proposition 1 *Let $M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_2)$ be two k-models. Then the following properties hold:*

(1) $\phi \in KM_1$ iff $W_1 \subseteq Mod(\phi)$;
(2) $W_1 \subseteq W_2$ iff $KM_2 \subseteq KM_1$;
(3) $KM_1 = KM_2$ iff $W_1 = W_2$;
(4) Let $M' = (W_1 \cup W_2, w)$, then $KM' = KM_1 \cap KM_2$;
(5) Let $w' \in W_1 \cap W_2$ and $M' = (W_1 \cap W_2, w')$, then $KM_1 \cup KM_2 \subseteq KM'$.

Readers are reminded that we may flexibly use Proposition 1 to move between sets of possible worlds and sets of formulas.

Given a set of k-models $\mathcal{S}$ and a k-model M, let $\leq_M$ be an ordering on $\mathcal{S}$ as we defined in Definition 2. By $Min(\mathcal{S}, \leq_M)$ we mean the set of all elements in $\mathcal{S}$ that are minimal with respect to ordering $\leq_M$. The following proposition simply shows that $\leq_M$ is a partial ordering.

Proposition 2 *Let M be a k-model. Then $\leq_M$ defined in Definition 2 is a partial ordering.*

The following proposition follows from Definitions 3 and 4.

Proposition 3 *Let T and μ be two formulas. Then $Mod(T \diamond \mu) = \bigcup_{M \in Mod(T)} Min(Mod(\mu), \leq_M)$.*

Proof: To prove the result, we only need to show that for each k-model M of T, $Res(M, \mu) = Min(Mod(\mu), \leq_M)$. Let $M' \in Res(M, \mu)$. Since $M' \models \mu$, $M' \in Mod(\mu)$. On the other hand, according to Definition 3, for any $M'' \in Mod(\mu)$, we have $M'' \not<_M M'$. That is, $M' \in Min(Mod(\mu), \leq_M)$. So $Res(M, \mu) \subseteq Min(Mod(\mu), \leq_M)$. Similarly, we can show

$Min(Mod(\mu), \leq_M) \subseteq Res(M, \mu)$. ■

The above proposition provides an important characterization on knowledge update in terms of a particular minimal change criterion. Now the question we are interested in is whether our knowledge update operator satisfies some classical properties of belief (knowledge base) update. In the last decade, belief update has been extensively studied by many researchers and its difference from belief revision is well understood [14,22,39]. From the observation of semantic difference between belief update and revision, Katsuno and Mendelzon [16] argued that the original revision postulates proposed by Gardenfors *el al.* [7] are not quite suitable for update, and ignoring such difference may lead to unreasonable solutions [16]. Instead, Katsuno and Mendelzon [16] proposed alternative postulates for any update operator $\diamond$ as follows.

(U1) $T \diamond \mu \models \mu$.
(U2) If $T \models \mu$ then $T \diamond \mu \equiv T$.
(U3) If both T and μ are satisfiable then $T \diamond \mu$ is also satisfiable.
(U4) If $T_1 \equiv T_2$ and $\mu_1 \equiv \mu_2$ then $T \diamond \mu_1 \equiv T_2 \diamond \mu_2$.
(U5) $(T \diamond \mu) \wedge \alpha \models T \diamond (\mu \wedge \alpha)$.
(U6) If $T \diamond \mu_1 \models \mu_2$ and $T \diamond \mu_2 \models \mu_1$ then $T \diamond \mu_1 \equiv T \diamond \mu_2$.
(U7) If T is complete (i.e., has a unique k-model) then $(T \diamond \mu_1) \wedge (T \diamond \mu_2) \models T \diamond (\mu_1 \vee \mu_2)$.
(U8) $(T_1 \vee T_2) \diamond \mu \equiv (T_1 \diamond \mu) \vee (T_2 \diamond \mu)$.

Under the context of S5 modal logic, we assume all the formulas occurring in the above postulates are S5 formulas. The following theorem shows that our knowledge update operator satisfies all these postulates.

Theorem 1 *Knowledge update operator $\diamond$ defined in Definition 4 satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).*

It is worth mentioning that in [16] Katsuno and Mendelzon relate their update postulates for belief updates to the ordering used in defining the updates. We can not use their result directly for the proof of the above theorem as their result pertains to propositional theories and ordering between propositional interpretations.

4 Characterizing specific knowledge updates

While the previous section studies general minimal change properties of our knowledge update, alternative characterizations of knowledge update can be described for several specific forms. These specific forms present important features of knowledge update, and their alternative characterizations are convenient when the use of the notion of knowledge update becomes an overkill. For example, the alternative characterization of *sensing update* below is a much simpler characterization that is used in reasoning about sensing actions [26,28].

4.1 Gaining knowledge update

We first introduce a notation that will be useful in our following discussions. Let W be a set of worlds and $w \in W$. By $W^{(w,\phi)}$, we denote the set $\{w' \mid w' \in W \text{ and } (w' \models \phi \text{ iff } w \models \phi)\ \}$.

Proposition 4 *Consider T and $K\phi$ where ϕ is objective and $T \models \phi$. Then*

(1) If $M' = (W', w')$ is a k-model of $T \diamond K\phi$, then there exists a k-model $M = (W, w)$ of T such that $w = w'$ and $W' = W^{(w,\phi)}$;
(2) If $M = (W, w)$ is a k-model of T, then $M' = (W^{(w,\phi)}, w)$ is a k-model of $T \diamond K\phi$.

The above proposition reveals an important property about knowledge update as observed by a reasoner [3]: to know some fact, the agent only needs to restrict the current possible worlds in each of her k-models, if this fact itself is already entailed by her current knowledge set. We call this kind of knowledge update *gaining knowledge update.*

Corollary 1 *For an objective formula ϕ, if $T \models \phi$ then $T \diamond K\phi \models \phi$.*

Example 4 *Let $T \equiv a \wedge \neg Ka \wedge Kb$. Suppose $w_0 = \{a, b\}$, $w_1 = \{a\}$, $w_2 = \{b\}$ and $w_3 = \emptyset$. Then T has one k-model $M = (\{w_0, w_2\}, w_0)$. Updating M with Ka, according to our k-model update definition, we have a unique resulting*

[3] Note that the update $T \diamond \mu$ is done by a third party; not the agent. For example, in the domain of an agent that needs to use a plan with sensing action (which will give him new knowledge), the planner or plan verifier is the third party which constructs the plan or verifies if the plan will indeed achieve the goal. In that case T expresses the state of the world from the third party's view point. For example if $T_1 = a$, then it means that a is true in the real world but our agent does not know it. On the other hand if $T_2 = Ka$, then it means that a is true in the real world and our agent knows it. Similarly, if $\mu = Ka \vee K\neg a$ is the effect of a sensing action, then the third party reasons that after executing that action (or after updating the initial theory by μ) the agent would know the value of a. In this case updating of T_1 with μ will result in T_2, and the third party will known that after the sensing action the agent would know that a is true; while the agent did not know that before the sensing action. We discuss this further in a later section.

k-model $M' = (\{w_0\}, w_0)$. Indeed, this result is also obtained from Proposition 4.

4.2 Ignorance update

As a contrary case to the gaining knowledge update, we now characterize an agent ignoring a fact from her knowledge set which we call *ignorance update*, i.e. updating T with $\neg K\phi$. From Definitions 3 and 4, it is easy to see that $T \diamond \neg\phi \models \neg K\phi$. However, it should be noted that $T \diamond \neg\phi$ can *not* be used to achieve $T \diamond \neg K\phi$. Consider a k-model $M = (\{\{a, b\}, \{a\}\}, \{a, b\})$. Updating M with $\neg Ka$ we have a possible resulting k-model $M' = (\{\{a, b\}, \{a\}, \{b\}\}, \{a, b\})$, while updating M with $\neg a$ will lead to a possible result $M'' = (\{\{a, b\}, \{a\}, \{b\}\}, \{b\})$. Note that both M' and M'' entail $\neg Ka$, but $M' <_M M''$ according to Definition 2.

Proposition 5 *Consider T and ϕ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond \neg K\phi$, then there exists a k-model $M = (W, w)$ of T such that
  - *(i) if $M \models K\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$,*
  - *(ii) otherwise, $w' = w$ and $W' = W$;*

(2) If $M = (W, w)$ is a k-model of T, then $M' = (W', w')$ is a k-model of $T \diamond \neg K\phi$, where
  - *(i) if $M \models K\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$,*
  - *(ii) otherwise, $w' = w$ and $W' = W$.*

Example 5 *Suppose $T \equiv \neg Ka \wedge \neg Kb \wedge K(a \vee b) \wedge Kc$ and the agent wants to ignore c. Let $w_0 = \{a, b, c\}$, $w_1 = \{a, c\}$, $w_2 = \{b, c\}$, $w_3 = \{c\}$, $w_4 = \{a, b\}$, $w_5 = \{a\}$, $w_6 = \{b\}$, $w_7 = \emptyset$. Clearly, T has three k-models: $M_0 = (\{w_0, w_1, w_2\}, w_0)$, $M_1 = (\{w_0, w_1, w_2\}, w_1)$, and $M_2 = (\{w_0, w_1, w_2\}, w_2)$. From Proposition 4, $T \diamond \neg Kc$ has the following twelve k-models: $(\{w_0, w_1, w_2, w_i\}, w_j)$, where $i = 4, 5, 6, 7$ and $j = 0, 1, 2$.*

4.3 Sensing update

Now we consider the case when μ is of the form $K\phi \vee K\neg\phi$ where ϕ is objective. Updating T with this type of μ is particularly useful in reasoning about sensing actions [26,28] where $K\phi \vee K\neg\phi$ represents the effect of a sensing action that senses ϕ. After the execution of such a sensing action an agent will know either ϕ or $\neg\phi$. We refer to such an update as a *sensing update*. The following proposition characterizes the update of T with a formula of the form $K\phi \vee$

$K\neg\phi$. It is interesting to note that the sufficient and necessary condition for a k-model of $T \diamond (K\phi \vee K\neg\phi)$ is similar to the one presented in Proposition 4.

Proposition 6 *Consider T and $\mu \equiv K\phi \vee K\neg\phi$ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond (K\phi \vee K\neg\phi)$, then there exists a k-model $M = (W, w)$ of T such that $w = w'$ and $W' = W^{(w,\phi)}$, or $w = w'$ and $W' = W^{(w,\neg\phi)}$;
(2) If $M = (W, w)$ is a k-model of T, then $M' = (W', w')$ is a k-model of $T \diamond (K\phi \vee K\neg\phi)$, where $w' = w$ and $W' = W^{(w,\phi)}$, or $w' = w$ and $W' = W^{(w,\neg\phi)}$.

The following corollary says that if ϕ is true in the real world then after sensing ϕ (i.e., doing an update with $K\phi \vee K\neg\phi$) an agent will know that ϕ is true.

Corollary 2 *For objective formulas ϕ, if $T \models \phi$ then $T \diamond (K\phi \vee K\neg\phi) \models K\phi$.*

Example 6 *Suppose $T \equiv Kb \wedge \neg Ka \wedge \neg K\neg a$ represents the current knowledge of an agent. Note that T implies that the agent does not have any knowledge about a. Consider the update of T with $\mu \equiv Ka \vee K\neg a$ which can be thought of as the agent trying to reason – in the planning or plan verification stage – about a sensing action*[4] *that will give her the knowledge about a. Let $w_0 = \{a, b\}, w_1 = \{b\}, w_2 = \{a\}$ and $w_3 = \emptyset$. It is easy to see that $M_0 = (\{w_0, w_1\}, w_0)$ and $M_1 = (\{w_0, w_1\}, w_1)$ are two k-models of T. Then according to the above proposition, it is obtained that $M'_0 = (\{w_0\}, w_0)$ and $M'_1 = (\{w_1\}, w_1)$ are the two k-models of $T \diamond \mu$.*

4.4 Forgetting update

We now consider another important type of knowledge update, the update of T with $\mu \equiv \neg K\phi \wedge \neg K\neg\phi$. This update can be thought of as the result of an agent *forgetting* her knowledge about the fact ϕ. We will refer to such an update as a *forgetting update*. The following proposition shows that in order to forget ϕ from T, for each k-model of the current knowledge set, the agent only needs to expand the set of possible worlds of this model with exactly *one specific* world.

Proposition 7 *Consider T and $\mu \equiv \neg K\phi \wedge \neg K\neg\phi$ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond \mu$, then there exists a k-model $M = (W, w)$ of T such that

[4] Such reasoning is necessary in creating plans with sensing actions or verifying such plans. On the other hand after the execution of a sensing action the agent exactly knows either a or $\neg a$, and can simply use the notion of belief update.

(i) if $M \models K\phi$*, then* $w' = w$ *and* $W' = W \cup \{w^*\}$*, where* $w^* \models \neg\phi$*;*
(ii) if $M \models K\neg\phi$*, then* $w' = w$ *and* $W' = W \cup \{w^*\}$*, where* $w^* \models \phi$*;*
(iii) otherwise, $w' = w$ *and* $W' = W$*.*

(2) If $M = (W, w)$ *is a k-model of* T*, then* $M' = (W', w)$ *is a k-model of* $T \diamond \mu$ *where*
(i) if $M \models K\phi$*, then* $W' = W \cup \{w^*\}$*, where* $w^* \models \neg\phi$*;*
(ii) if $M \models K\neg\phi$*, then* $W' = W \cup \{w^*\}$*, where* $w^* \models \phi$*;*
(iii) otherwise, $W' = W$*.*

Example 7 *Suppose* $T \equiv Kb \wedge (Ka \vee K\neg a)$ *represents the current knowledge of an agent. After executing a forgetting action the agent now would like to update her knowledge with* $\mu \equiv \neg Ka \wedge \neg K\neg a$*. Let* $w_0 = \{a, b\}, w_1 = \{b\}, w_2 = \{a\}, w_3 = \emptyset$*. It is easy to see that* $M_0 = (\{w_0\}, w_0)$ *and* $M_1 = (\{w_1\}, w_1)$ *are the two k-models of* T*. Then using Proposition 7, we conclude that* $M_0' = (\{w_0, w_1\}, w_0)$*,* $M_1' = (\{w_0, w_3\}, w_0)$*,* $M_2' = (\{w_1, w_0\}, w_1)$*, and* $M_3' = (\{w_1, w_2\}, w_1)$ *are the four k-models of* $T \diamond \mu$*. Note that* $(\{w_0, w_2\}, w_0)$ *cannot be a k-model of* $T \diamond \mu$ *according to Proposition 7.*

5 Persistence of knowledge and ignorance

Like most systems that do dynamic modeling, knowledge update discussed previously is non-monotonic in the sense that while adding new knowledge into a knowledge set, some previous knowledge in the set might be lost. However, it is important to investigate classes of formulas that are persistent with respect to an update, as this may partially simplify the underlying inference problem [38]. Furthermore, characterizing persistence is also an important issue in non-monotonic epistemic logic reasoning because it plays an essential role in the way of how different states of agent's knowledge can be compared [5,31,32].

Given T and μ, a formula α is said to be *persistent* with respect to the update of T with μ, if $T \models \alpha$ implies $T \diamond \mu \models \alpha$. If α is of the form $K\phi$, we call this persistence as *knowledge persistence*, while if α is of the form $\neg K\phi$, we call it *ignorance persistence*. The question that we address now is that under what conditions, a formula α is persistent with respect to the update of T with μ.

As the update of T with μ is achieved based on the update of every k-model of T with μ, our task reduces to the study of persistence with respect to a k-model update. This is defined in the following definition.

Definition 5 *(***Persistence with respect to** k**-model update***) Let* μ *and* α *be two formulas and* M *be a k-model.* α *is* persistent *with respect to the update of* M *with* μ *if for any* $M' \in Res(M, \mu)$*,* $M \models \alpha$ *implies* $M' \models \alpha$*.*

Clearly a formula α is persistent with respect to the update of T with μ if and only if for each k-model M of T, α is persistent with respect to the update of M with μ. To characterize the persistence property with respect to k-model updates, we first define a preference ordering on k-models in terms of a formula.

Definition 6 *(***Formula closeness***) Let μ be a formula and M_1 and M_2 be two k-models. We say that M_1 is* as close *to μ as M_2, denoted as $M_1 \leq_\mu M_2$, if one of the following conditions holds:*

(1) $M_1 \in Mod(\mu)$;
(2) $M_1, M_2 \notin Mod(\mu)$, and for any $M \in Mod(\mu)$, $M_1 \leq_M M_2$.

We denote $M_1 <_\mu M_2$ if $M_1 \leq_\mu M_2$ and $M_2 \not\leq_\mu M_1$.

Intuitively, the above definition specifies a partial ordering to measure the closeness between two k-models to a formula. In particular, if M_1 is a k-model of μ, then M_1 is closer to μ than all other k-models (i.e. condition 1). If neither M_1 nor M_2 is a k-model of μ, then the comparison between M_1 and M_2 with respect to μ is defined based on the k-model preference ordering $\leq_M$ for each k-model M of μ (i.e. condition 2). Note that if both M_1 and M_2 are k-models of μ, we have $M_1 \leq_\mu M_2$ and $M_2 \leq_\mu M_1$, and both of them are equally close to μ.

Example 8 *Let $\mu \equiv Ka \wedge Kb$, $w_0 = \{a, b\}$, $w_1 = \{b\}$, $w_2 = \{a\}$ and $w_3 = \emptyset$. Clearly, μ has one k-model $M = (\{w_0\}, w_0)$. Consider two k-models $M_1 = (\{w_0, w_1\}, w_0)$ and $M_2 = (\{w_1, w_2\}, w_1)$. Now let us compare which one of them is closer to μ. Since neither M_1 nor M_2 is a k-model of μ, we can use condition 2 in Definition 6 to compare M_1 and M_2. According to Definition 2, it is easy to see that $M_1 \leq_M M_2$ as $w_0 \setminus w_1 \cup w_1 \setminus w_0 = \{a\} \neq \emptyset$. Therefore, we conclude $M_1 \leq_\mu M_2$. Furthermore, we also have $M_1 <_\mu M_2$.*

Proposition 8 *Let μ be a formula. For any two k-models M_1 and M_2, if $M_1 \leq_\mu M_2$, then $M_2 \models \mu$ implies $M_1 \models \mu$.*

Proof: Suppose $M_2 \models \mu$. Then $M_2 \in Mod(\mu)$. From Definition 6, we know that for any other k-model M', $M_2 \leq_\mu M'$. So $M_2 \leq_\mu M_1$. But we have $M_1 \leq_\mu M_2$. This implies that both M_1 and M_2 are equally close to μ. Hence, $M_1 \models \mu$. ■

Given a formula μ and a sequence of k-models $M_1, \cdots, M_k$, if the relation $M_1 \leq_\mu M_2 \leq_\mu \cdots \leq_\mu M_k$ holds, then it means that M_i is closer to μ than M_j, where $i < j$. Now under this condition, if there is another formula α which satisfies the property that $M_j \models \alpha$ implies $M_i \models \alpha$ whenever $i < j$, we say that formula α is *persistent* with respect to formula μ. In other words, when

k-models move closer to μ, α's truth value is preserved in these k-models. The following definition formalizes this idea.

Definition 7 *(**$\leq_\mu$-persistence**) Let α, μ be two formulas. We say that α is* $\leq_\mu$-persistent *if for any two k-models M_1 and M_2, $M_2 \models \alpha$ and $M_1 \leq_\mu M_2$ implies $M_1 \models \alpha$.*

Now we have the following important relationship between $\leq_\mu$-persistence and k-model update persistence.

Theorem 2 *Let α and μ be two formulas and M be a k-model. α is persistent with respect to the update of M with μ if α is $\leq_\mu$-persistent.*

Proof: Let M' be a k-model in $Res(M, \mu)$. Then we have $M' \in Mod(\mu)$. So for any k-model M'', we have $M' \leq_\mu M''$. So $M' \leq_\mu M$. Now suppose α is μ-persistent. It follows that $M \models \alpha$ implies $M' \models \alpha$. As M' is an arbitrary k-model in $Res(M, \mu)$, we can conclude that α is persistent with respect to the update of M with μ. ■

From Theorem 2, we have that $\leq_\mu$-persistence is a sufficient condition to guarantee a formula's persistence with respect to a k-model update. As will be shown next, we can provide a unique characterization for μ-persistence. We first define the notion of ordering preservation as follows.

Definition 8 *(**Ordering Preservation**) Given two formulas α and β. We say that ordering $\leq_\alpha$* preserves *ordering $\leq_\beta$ if for any two k-models M_1 and M_2, $M_1 \leq_\alpha M_2$ implies $M_1 \leq_\beta M_2$.*

The intuition behind ordering preservation is clear. That is, if $\leq_\alpha$ preserves $\leq_\beta$, then for any two k-models M_1 and M_2, whenever M_1 is closer to α than M_2, M_1 will be closer to β than M_2 as well. Finally, we have the following important result to characterize μ-persistence.

Theorem 3 *Given two formulas α and μ, α is $\leq_\mu$-persistent if and only if $\leq_\mu$ preserves $\leq_\alpha$.*

Proof: ($\Rightarrow$) Suppose α is $\leq_\mu$-persistent. That is, for any two k-models M_1 and M_2, $M_1 \leq_\mu M_2$ and $M_2 \models \alpha$ implies $M_1 \models \alpha$. So under the constraint that α is μ-persistent, whenever $M_1 \leq_\mu M_2$, we have $M_1 \leq_\alpha M_2$. That means, $\leq_\mu$ preserves $\leq_\alpha$.

($\Leftarrow$) Suppose $\leq_\mu$ preserves $\leq_\alpha$. From Definition 8, we have that for any two k-models M_1 and M_2, $M_1 \leq_\mu M_2$ implies $M_1 \leq_\alpha M_2$. Now suppose $M_1 \leq_\mu M_2$. So we have $M_1 \leq_\alpha M_2$. From Proposition 7, we have that $M_2 \models \alpha$ implies $M_1 \models \alpha$. From this it follows that α is $\leq_\mu$-persistent. ■

6 Background on computational complexity

In the rest of this paper, we consider complexity issues of knowledge update. In particular, we investigate the computational complexity of model checking for knowledge update.

We first introduce basic notions from complexity theory and refer to [8] for further details. Two important complexity classes are P and NP. The class of P includes those decision problems solvable by a polynomial-time deterministic Turing machine. The class of NP, on the other hand, consists of those decision problems solvable by a polynomial-time nondeterministic Turing machine.

Let $\mathcal{C}$ be a class of decision problems. The class $P^{\mathcal{C}}$ consists of the problems solvable by a polynomial-time deterministic Truing machine with an oracle for a problem from $\mathcal{C}$, while the class $NP^{\mathcal{C}}$ includes the problems solvable by a nondeterministic Turing machine with an oracle for a problem in $\mathcal{C}$. By co-$\mathcal{C}$ we mean the class consisting of the complements of the problems in $\mathcal{C}$.

The classes Σ_k^P and Π_k^P of the *polynomial hierarchy* are defined as follows:

$\Sigma_0^P = \Pi_0^P = P$, and
$\Sigma_k^P = NP^{\Sigma_{k-1}^P}$, $\Pi_k^P =$co-Σ_k^P for all $k > 1$.

It is easy to see that $NP = \Sigma_1^P$ and co-$NP = \Pi_1^P$. A problem A is *complete* for a class $\mathcal{C}$ if $A \in \mathcal{C}$ and for every problem B in $\mathcal{C}$ there is a polynomial transformation of B to A.

The prototypical Σ_k^P-complete and Π_k^P-complete problems are deciding the validity of quantified Boolean formulas (QBFs) of the form:

$$Q_1 X_1 Q_2 X_2 \cdots Q_k X_k E, k \geq 1, \tag{1}$$

where E is a Boolean expression using propositional atoms over alphabets $X_1, X_2, \cdots$, and X_k, and the Q_i's are alternating qualifiers from $\{\forall, \exists\}$ $(1 \leq i \leq k)$. If $Q_1 = \exists$, then deciding the validity of (1) is Σ_k^P-complete, while deciding the validity of (1) is Π_k^P-complete if $Q_1 = \forall$.

Let X and Y be two finite set of propositional atoms where X and Y have the same cardinality, i.e. $|X| = |Y|$. For convenience, we use notions $X \equiv Y$ to stand for formula $(x_1 \equiv y_1) \wedge (x_2 \equiv y_2) \wedge \cdots \wedge (x_m \equiv y_m)$. Consequently, $X \equiv \neg Y$ stands for formula $(x_1 \equiv \neg y_1) \wedge$ $(x_2 \equiv \neg y_2) \wedge \cdots \wedge (x_m \equiv \neg y_m)$. We also use $\neg X$ to denote the set $\{\neg x_i \mid x_i \in X\}$ (or formula $\bigwedge_{x_i \in X} \neg x_i$), and use notion $\bigvee \neg X$ to stand for formula $\bigvee_{x_i \in X} \neg x_i$. For a given formula α, we use $|\alpha|$ to denote the length of α.

The problem of *model checking* for knowledge update is described as follows: Given a knowledge set T, a formula μ, and a k-model M, deciding whether $M \in Mod(T \diamond \mu)$. It is well known that the model checking problem for traditional belief revision and update is located at the lower end of the polynomial hierarchy from P to Σ_2^P depending on specific revision/update operators and additional restrictions (if any) [19].

7 Complexity of model checking: General case

In this section, we investigate the complexity of model checking for the general case of knowledge update. When we say complexity of model checking we mean the complexity of checking whether $M \in Mod(T \diamond \mu)$ with respect to the size of M, T and μ. In this we assume that the representation of k-model M is such that all k-models need the same (or at most polynomial in the size of M) number of bits for representation.

Lemma 1 *Let $M = (W, w), M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_2)$ be three k-models.*

(1) Deciding whether $KM \setminus KM_2 \neq \emptyset$ and $KM_2 \setminus KM \neq \emptyset$ has time complexity $\mathcal{O}(|W| \times |W_2|)$.
(2) Deciding whether $KM \setminus KM_1 \subseteq KM \setminus KM_2$ has time complexity $\mathcal{O}(|W_1| \times (|W| + |W_2|))$.
(3) Deciding whether $KM_1 \setminus KM \subseteq KM_2 \setminus KM$ has time complexity $\mathcal{O}(|W_1| \times |W| \times |W_2|)$.

Lemma 2 *Let M, M_1 and M_2 be three k-models. Deciding whether $M_1 \leq_M M_2$ can be achieved in polynomial time in the size of the input: M, M_1 and M_2.*

Proof: According to Definition 2, if $w_1 \neq w_2$, then $M_1 \leq_M M_2$ iff $(w_1 \setminus w \cup w \setminus w_1) \subseteq (w_2 \setminus w \cup w \setminus w_2)$. Clearly, this can be verified in polynomial time. If $w_1 = w_2$, then we need to check the following conditions: (i) If $W \subseteq W_1$, then $M_1 \leq_M M_2$ iff condition (a) or (b) in Definition 2 is satisfied. From Lemma 1, we know that deciding whether (a) and (b) are true can be done in polynomial time. (ii) If $W_1 \subseteq W$, then $M_1 \leq_M M_2$ iff condition (a) or (c) in Definition 2 is satisfied. From Lemma 1, deciding whether (c) is true is in P. (iii) If $W \not\subset W_1$ and $W_1 \not\subset W$, then $M_1 \leq_M M_2$ iff conditions (b) and (c) should be satisfied. Again, deciding whether condition (c) is true is in polynomial time. So, the problem is in P. ■

Lemma 3 *Let M, M' be two k-models and μ a S5 formula. Deciding whether $M' \in Res(M, \mu)$ is in co-NP (in terms of the size of M, M' and μ).*

Proof: According to Proposition 3, if $M' \notin Res(M, \mu)$, there must exist another k-model M'' such that $M'' <_M M'$. A guess of a k-model M'' can be done in polynomial time. From Lemma 2, deciding whether $M'' \leq_M M'$ is in P (with respect to the size of M, M' and M''). Since $M'' <_M M'$ iff $M'' \leq_M M'$ and $M' \not\leq_M M''$, and since we assume that in our representation of k-models, all k-models need same number of bits for representation, it follows that checking whether $M'' <_M M'$ can be decided in polynomial time. So the problem is in co-NP. ■

Theorem 4 *Model checking for knowledge update is in Σ_2^P.*

Proof: From Definition 4, $M \in Mod(T \diamond \mu)$ iff for some $M' \in Mod(T)$, $M \in Res(M', \mu)$. A guess of M' and check whether $M' \in Mod(T)$, i.e. $M' \models T$, can be achieved in polynomial time. According to Lemma 3, deciding whether $M \in Res(M', \mu)$ can be solved with one call to a co-NP oracle. So the problem is in Σ_2^P. ■

The above result shows that model checking for knowledge update is in the same layer of the polynomial hierarchy as the traditional model based belief update. It should be noted though that the size of the input for model checking in knowledge updates is much larger from the input for model checking in belief updates [19].

7.1 Knowledge gradual update

The hardness can be simply proved by reducing model checking for Winslett's update operator [37] to our knowledge update operator, then following Liberatore and Schaerf's result [19], the hardness follows.

However, here we will present a different hardness proof because our proof gives rise a new subclass of knowledge update problems (yet different from Winslett's belief update) which can be viewed as a lower bound for knowledge update problems that are Σ_2^P-complete for model checking (see sections 8 and 9 for other subclasses of knowledge update problems).

Given T and μ, we say the update of T with μ is *knowledge gradual* if for any

k-model $M' = (W', w')$ of $T \diamond \mu$, there exists a k-model $M = (W, w)$ of T such that either $W \subseteq W'$ or $W' \subseteq W$. Note that, after performing a knowledge gradual update, the agent's knowledge may be decreased or increased (or without change), and the agent's actual world may be changed as well.

Example 9 *Let $T = a \wedge \neg Ka$ and $\mu = K\neg a$. Obviously, T has a unique k-model $M = (\{\{a\}, \emptyset\}, \{a\})$. Then updating M with μ generates a unique k-model of $T \diamond \mu$: $M' = (\{\emptyset\}, \emptyset)$. Obviously, M' has increased knowledge from M and the actual world of M' is also different from M's.*

Theorem 5 *Model checking for knowledge update is Σ_2^P-complete. The hardness holds even if the update is knowledge gradual.*

8 A tractable subclass - knowledge decreased update

In this section, we identify a subclass of knowledge update problems for which model checking can be achieved in polynomial time. We first introduce a useful notation. Let α be a S5 formula and ϕ^α be an objective formula (i.e. no K occurs in it) occurring in α. We then say ϕ^α is an *objective sub-formula* of α. We denote the set of all objective sub-formulas of α as $Sub^o(\alpha)$. For instance, given $\alpha = Ka \vee K\neg b$, $Sub^o(\alpha) = \{a, b, \neg b\}$.

Definition 9 *Given S5 formulas T and μ, updating T with μ is called* knowledge decreased *if for any k-model $M' = (W', w')$ of $T \diamond \mu$, there exists a k-model $M = (W, w)$ of T such that (i) $W \subseteq W'$ and $w = w'$; and (ii) there exists some $\phi^\mu \in Sub^o(\mu)$, such that $W = \{w^* \mid w^* \in W' \text{ and } w^* \models \phi^\mu\}$ or $W = \{w^* \mid w^* \in W' \text{ and } w^* \models \neg\phi^\mu\}$.*

¿From the above definition, it is easy to see that if an update is knowledge decreased, then the actual world of the agent's state will not change, and the agent's knowledge can only be decreased. Furthermore, the set of possible worlds in the agent's original state can be specifically computed from her resulting state. This feature leads to a tractable result on the model checking for knowledge decreased update.

Theorem 6 *Model checking for knowledge decreased update can be achieved in polynomial time.*

Proof: Given T, μ and a k-model $M' = (W', w')$. Suppose $T \diamond \mu$ be knowledge decreased. To check whether $M' \in Mod(T \diamond \mu)$, we need to do the following things:

(1) Check whether $M' \models \mu$,

(2) Compute a subset W of W' such that for any $w^* \in W'$, $w^* \in W$ iff $w^* \models \phi^\mu$ or $w^* \models \neg\phi^\mu$ for some $\phi^\mu \in Sub^o(\mu)$,
(3) Check whether $(W, w) \models T$.

Clearly, Steps 1 and 3 can be done in polynomial time. As $|Sub^o(\mu)| \leq |\mu|$, it follows that Step 2 can be also done in polynomial time. ■

It is worthwhile to mention specific forms of knowledge decreased update which, as we have presented earlier, have important applications in practical domains.

Theorem 7 *Ignorance and forgetting updates are knowledge decreased.*

Proof: The proof directly follows from Propositions 5 and 7 respectively. ■

Corollary 3 *Model checking for ignorance and forgetting updates can be achieved in polynomial time.*

9 An intractable subclass - Knowledge increased update

In this section, we address another subclass of knowledge update problems whose model checking complexity are intractable but lower than the general case. Such investigation will be useful for us to design efficient model checking algorithms for these subclasses of update problems.

As a contrary case to the knowledge decreased update, the knowledge increased update is defined as follows.

Definition 10 *Given T and μ, updating T with μ is called* knowledge increased *if for any k-model $M' = (W', w')$ of $T \diamond \mu$, there exists a k-model $M = (W, w)$ of T such that (i) $W' \subseteq W$, and $w = w'$; and (ii) there exists some $\phi^\mu \in Sub^o(\mu)$, such that $W' = \{w^* \mid w^* \in W$ and $w^* \models \phi^\mu\}$ or $W' = \{w^* \mid w^* \in W$ and $w^* \models \neg\phi^\mu\}$.*

It is clear that if a knowledge increased update is performed to an agent's knowledge set, it only increases the agent's knowledge and does not change the agent's actual world. Unfortunately, different from the knowledge decreased update, the model checking problem for knowledge increased update is not tractable.

Theorem 8 *Model checking for knowledge increased update is NP-complete.*

It is interesting to note that some specific forms of knowledge update we discussed earlier such as gaining knowledge and sensing updates are knowledge increased.

Theorem 9 *Gaining knowledge and sensing updates are knowledge increased.*

Proof: The proof directly follows from Propositions 4 and 6 respectively. ■

Corollary 4 *Model checking for gaining knowledge and sensing updates are NP-complete.*

10 Conclusions

In this paper we developed an explicit notion of knowledge update as an analogous notion to belief update and illustrated its usefulness in characterizing the knowledge change of an agent in presence of new knowledge. In our formulation, knowledge update is particularly relevant in reasoning about actions and plan verifications when there are sensing or forgetting actions. We presented simpler alternative characterization of knowledge update for particular cases, and showed its equivalence to the original characterization. We discussed when particular knowledge (or ignorance) persists with respect to a knowledge update. We also undertook a further study about the complexity issue of knowledge update. In particular, we analyzed the complexity of model checking for knowledge update in the general case and in special cases. We identify special subcases where the model checking is either tractable or its complexity is lower than the general case. We expect that these results will be useful for designing more optimal model checking algorithms in the implementation of knowledge update.

We believe our work here to be a starting point on knowledge update, and as evident from the research in belief update and revision in the past decade, a lot needs to be done in knowledge update. For example, issues such as multi-agent knowledge update, iterative knowledge update, abductive knowledge update, minimal knowledge in knowledge update, etc. remain to be explored. Similarly, in regards to reasoning about actions, additional specific cases of knowledge update need to be identified and simpler alternative characterization for them need to be developed. On the other hand, as our knowledge update is developed based on Kripke models, it may be integrated into model checking formalism so that a unified system of model checking and model updating can be used not only for automatic system verification but also for automatic system modification.

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Appendix: Proofs

Proposition 1 *Let $M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_2)$ be two k-models. Then the following properties hold:*

(1) $\phi \in KM_1$ iff $W_1 \subseteq Mod(\phi)$;
(2) $W_1 \subseteq W_2$ iff $KM_2 \subseteq KM_1$;
(3) $KM_1 = KM_2$ iff $W_1 = W_2$;
(4) Let $M' = (W_1 \cup W_2, w)$, then $KM' = KM_1 \cap KM_2$;
(5) Let $w' \in W_1 \cap W_2$ and $M' = (W_1 \cap W_2, w')$, then $KM_1 \cup KM_2 \subseteq KM'$.

Note that under the finite model restriction, the above results are simply those statements presented in Exercise 3.1.7.1, Exercise 3.1.7.2 and Exercise 3.1.6.2 of [23]. For readers' convenience, here we still present our complete proof as follows.

Proof: (1). ($\Rightarrow$) From $\phi \in KM_1$, we have for all $w' \in W_1$, $w' \models \phi$ (note that ϕ is a formula without containing modal operator K). That is, $w' \in Mod(\phi)$. So $W_1 \subseteq Mod(\phi)$.

($\Leftarrow$) Suppose $W_1 \subseteq Mod(\phi)$. Then we have for any $w' \in W_1$, $w' \models \phi$. That is, $\phi \in KM_1$.

(2). Let $\bigwedge KM_1$ and $\bigwedge KM_2$ be the conjunctions of all prime formulas in KM_1 and KM_2 respectively. Then it is clear that $Mod(\bigwedge KM_1) = W_1$ and $Mod(\bigwedge KM_2) = W_2$. So we have $KM_2 \subseteq KM_1$ iff $Mod(\bigwedge KM_1) \subseteq Mod(\bigwedge KM_2)$ iff $W_1 \subseteq W_2$.

(3). In the proof of 2, we stated that $Mod(\bigwedge KM_1) = W_1$ and $Mod(\bigwedge KM_2) = W_2$. So $KM_1 = KM_2$ iff $Mod(\bigwedge KM_1) = Mod(\bigwedge KM_2)$ iff $W_1 = W_2$.

(4). According to the definition of M', we have $\phi \in KM'$ iff for all $w' \in W_1 \cup W_2$, $w' \models \phi$ iff for all $w' \in W_1$, $w' \models \phi$ and for all $w'' \in W_2$, $w'' \models \phi$ iff $\phi \in KM_1$ and $\phi \in MK_2$ iff $\phi \in KM_1 \cap KM_2$.

(5). If $\phi \in KM_1 \cup KM_2$, we have $\phi \in KM_1$ or $\phi \in KM_2$. So either for all $w_1 \in W_1$, we have $w_1 \models \phi$ or for all $w_2 \in W_2$, we have $w_2 \models \phi$. In either case, for any $w' \in W_1 \cap W_2$, we have $w' \models \phi$. That is, $\phi \in KM'$. ■

Proposition 2 *Let M be a k-model. Then $\leq_M$ defined in Definition 2 is a partial ordering.*

Proof: From Definition 2, it is clear that $\leq_M$ is reflexive and antisymmetric. Now we prove $\leq_M$ is also transitive. Let $M = (W, w)$, $M_1 = (W_1, w_1)$,

$M_2 = (W_2, w_2)$ and $M_3 = (W_3, w_3)$ be k-models, and $M_1 \leq_M M_2$ and $M_2 \leq_M M_3$. Now we prove $M_1 \leq_M M_3$.
Case 1. Suppose $M_1 \leq_M M_2$ is due to condition (1) in Definition 2, i.e. $(w_1 \setminus w \cup w \setminus w_1) \subset (w_2 \setminus w \cup w \setminus w_2)$. Consider $M_2 \leq_M M_3$. According to Definition 2, either condition (1) or (2) is satisfied. If condition (1) is satisfied, then $(w_2 \setminus w \cup w \setminus w_2) \subset (w_3 \setminus w \cup w \setminus w_3)$. This follows $(w_1 \setminus w \cup w \setminus w_1) \subset (w_3 \setminus w \cup w \setminus w_3)$. So $M_1 \leq_M M_3$. If condition (2) is satisfied, it means $w_2 = w_3$, it also follows $(w_1 \setminus w \cup w \setminus w_1) \subset (w_3 \setminus w \cup w \setminus w_3)$, and therefore $M_1 \leq_M M_3$.
Case 2. Now suppose $M_1 \leq_M M_2$ is due to condition (2) in Definition 2, i.e. $w_1 = w_2$ and one of conditions (i), (ii), (iii), (iv), or (v) is satisfied. If $M_2 \leq_M M_3$ is due to condition (1) in Definition 2, i.e. $(w_2 \setminus w \cup w \setminus w_2) \subset (w_3 \setminus w \cup w \setminus w_3)$, it follows that $M_1 \leq_M M_3$ because $w_1 = w_2$. Suppose $M_2 \leq_M M_3$ is due to condition (2) in Definition 2, that is, $w_2 = w_3$ and one of conditions (i), (ii), (iii), (iv), or (v) is satisfied. Here we only consider the following three cases, while all other cases can be proved in a similar way.
Case 2.1. Both $M_1 \leq_M M_2$ and $M_2 \leq_M M_3$ are due to condition (2) and (ii) in Definition 2. Under this case, we can only have (a) $KM_3 \subset KM_2 \subset KM_1 \subset KM$; or (b) $KM_2 \subset KM_1 \subset KM$ but $KM \setminus KM_2 \neq \emptyset$ and $KM_2 \setminus KM \neq \emptyset$. Clearly, in either case, we have $M_1 \leq_M M_3$.
Case 2.2. $M_1 \leq_M M_2$ is due to condition (2) and (ii) and $M_2 \leq_M M_3$ are due to condition (2) and (iii) in Definition 2. By analyzing Definition 2, it concludes that this situation will never occur. This is because from $M_1 \leq_M M_2$, we can only have either $KM_2 \subset KM$ or $KM \setminus KM_2 \neq \emptyset$ and $KM_2 \setminus KM \neq \emptyset$, and from $M_2 \leq_M M_3$, we can only have $KM \subset KM_2$. Obviously, these two cases conflict with each other.
Case 2.3. $M_1 \leq_M M_2$ is due to condition (2) and (ii) and $M_2 \leq_M M_3$ are due to condition (2) and (iv) in Definition 2. Using (2)(ii), we will have $KM_1 \subset KM$ and $KM_2 \setminus KM \neq \emptyset$ and $KM \setminus KM_2 \neq \emptyset$. Using (2)(iv) we have $KM_2 \setminus KM \cup KM \setminus KM_2 \subseteq KM_3 \setminus KM \cup KM \setminus KM_3$. Thus we have $KM_1 \subset KM$ and $KM_3 \setminus KM \neq \emptyset$ and $KM \setminus KM_3 \neq \emptyset$. This implies $M_1 \leq_M M_3$. ■

Theorem 1 *Knowledge update operator $\diamond$ defined in Definition 4 satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).*

Proof: From Definitions 3 and 4, it is easy to verify $\diamond$ satisfies postulates (U1)-(U4). For illustration purposes we give the proof of (U1). Let (W, w) be an arbitrary k-model of $T \diamond \mu$. To show (U1) we need to show that $(W, w) \models \mu$. By Definition 4, there must exist a model $M = (W', w')$ of T such that $(W, w) \in Res(M, \mu)$. By Definition 3, for (W, w) to be in $Res(M, \mu)$, it must be the case that $(W, w) \models \mu$.

Now we prove $\diamond$ satisfies (U5). To prove that $(T \diamond \mu) \wedge \alpha \models T \diamond (\mu \wedge \alpha)$, it is sufficient to prove that for each k-model of T, say M, $Res(M, \mu) \cap Mod(\alpha) \subseteq$

$Res(M, \mu \wedge \alpha)$. In particular, we need to show for any $M' \in Res(M, \mu) \cap Mod(\alpha)$, $M' \in Res(M, \mu \wedge \alpha)$. Suppose $M' \notin Res(M, \mu \wedge \alpha)$. According to Definition 3, we have (1) $M' \not\models \mu \wedge \alpha$; or (2) there exists another k-model M'' such that $M'' \models \mu \wedge \alpha$ and $M'' <_M M'$. If it is case (1), it follows that $M' \notin Res(M, \mu) \cap Mod(\alpha)$. Then the result holds. If it is case (2), it also implies that $M'' \models \mu$ and $M'' <_M M'$. That means, $M' \notin Res(M, \mu)$ from Definition 3. The result still holds.

Now we prove $\diamond$ satisfies (U6). Similarly, to prove $\diamond$ satisfies (U6), we only need to prove for any k-model of T, say M, if $Res(M, \mu_1) \subseteq Mod(\mu_2)$ and $Res(M, \mu_2) \subseteq Mod(\mu_1)$, then $Res(M, \mu_1) = Res(M, \mu_2)$. We first prove $Res(M, \mu_1) \subseteq Res(M, \mu_2)$. Let $M' \in Res(M, \mu_1)$. Then $M' \models \mu_2$. Suppose $M' \notin Res(M, \mu_2)$. It follows that there exists another $M'' \in Res(M, \mu_2)$ such that $M'' <_M M'$. Also note that $M'' \models \mu_1$. This contradicts the fact that $M' \in Res(M, \mu_1)$. This proves $Res(M, \mu_1) \subseteq Res(M, \mu_2)$. Similarly, we can prove $Res(M, \mu_2) \subseteq Res(M, \mu_1)$.

Now we prove $\diamond$ satisfies (U7). Since T is complete, it follows that T has a unique k-model M. So we only need to prove $Res(M, \mu_1) \cap Res(M, \mu_2) \subseteq Res(M, \mu_1 \vee \mu_2)$. Let $M' \in Res(M, \mu_1) \cap Res(M, \mu_2)$. Suppose $M' \notin Res(M, \mu_1 \vee \mu_2)$. Then there exists a k-model $M'' \in Res(M, \mu_1 \vee \mu_2)$ such that $M'' <_M M'$. Note that $M'' \models \mu_1 \vee \mu_2$. If $M'' \models \mu_1$, it will follow that $M' \notin Res(Mu, \mu_1)$, otherwise, $M' \notin Res(Mu, \mu_2)$. In either case, we have $M' \notin Res(M, \mu_1) \cap Res(M, \mu_2)$. This proves the result.

Finally, the fact that $\diamond$ satisfies (U8) is obtained straightforward from Definitions 3 and 4. ■

Proposition 4 *Consider T and ϕ where ϕ is objective and $T \models \phi$. Then*

(1) If $M' = (W', w')$ is a k-model of $T \diamond K\phi$, then there exists a k-model $M = (W, w)$ of T such that $w = w'$ and $W' = W^{(w,\phi)}$;
(2) If $M = (W, w)$ is a k-model of T, then $M' = (W^{(w,\phi)}, w)$ is a k-model of $T \diamond K\phi$.

Proof: To prove this proposition, we first prove the following result:

> Consider T and $K\phi$ where ϕ is objective and $T \models \phi$. Let $M = (W, w)$ be a k-model of T and $M' = (W^{(w,\phi)}, w)$. For any $M'' \in Mod(K\phi)$ and $M'' \neq M'$, $M' <_M M''$.

It is easy to see that for any $M'' = (W'', w'') \in Mod(K\phi)$, where $w'' \neq w$, $M' <_M M''$. Now let us consider $M'' = (W'', w)$, where $W'' \neq W^{(w,\phi)}$. We consider the following possible cases.

Case 1. $W'' \subset W^{(w,\phi)}$ (proper set inclusion). Since $W^{(w,\phi)} \subseteq W$, from Proposition 3, we have $KM \subseteq KM' \subset KM''$, and hence $KM' \setminus KM \subset KM'' \setminus KM$. From Definition 2, it follows $M' \leq_M M''$ and $M'' \not\leq_M M'$, that is, $M' <_M M''$.
Case 2. $W^{(w,\phi)} \subset W''$ (proper set inclusion). Without loss of generality, we assume $W'' = W^{(w,\phi)} \cup \{w_i\}$, where $w_i \models \phi$. Clearly, $w_i \notin W$ otherwise we will have $w_i \in W^{(w,\phi)}$ and then $W'' = W^{(w,\phi)}$. Since $W'' \not\subseteq W$ and $W \not\subseteq W''$, from Proposition 3, we have $KM \not\subseteq KM''$ and $KM' \not\subseteq KM$. Then it must be the case that $KM \setminus KM'' \neq \emptyset$ and $KM'' \setminus KM \neq \emptyset$. From Definition 2 (i.e. (iii) in condition 2), we know that $M' \leq_M M''$ and $M'' \not\leq_M M'$.
Case 3. $W^{(w,\phi)} \not\subset W''$ and $W'' \not\subset W^{(w,\phi)}$. Without loss of generality, we can assume that $W'' = W^{(w,\phi)} \cup \{w_i\} \setminus \{w_j\}$, where $w_j \in W^{(w,\phi)}$. Since we require that $M'' \models K\phi$, it follows that $w_i \models \phi$. Also, from the construction of $W^{(w,\phi)}$, we know that $w_i \notin W$ otherwise it reduces to the case that $W'' \subseteq W^{(w,\phi)}$. Therefore, $W'' \not\subseteq W$ and $W \not\subseteq W''$. From the above discussion, it follows that $KM \setminus KM'' \neq \emptyset$ and $KM'' \setminus KM \neq \emptyset$. So from Definition 2, we know that $M' \leq_M M''$ and $M'' \not\leq_M M'$.

Now by using the above result, we prove statements (1) and (2).

Proof of (1). $M' = (W', w')$ is a k-model of $T \diamond K\phi$ iff $M' \in Mod(T \diamond K\phi)$ iff there exists a k-model M of T, such that $M' \in Res(M, K\phi)$ iff $M' \in Min(Mod(K\phi), \leq_M)$ for some $M \in Mod(T)$. We now argue that this implies $W' = W^{(w,\phi)}$ and $w' = w$. Suppose this is not the case. Then let $M^* = (W^{(w,\phi)}, w)$. By the above result we then have $M^* <_M M'$. But this contradicts with $M' \in Min(Mod(K\phi), \leq_M)$. Hence our assumption is wrong and $W' = W^{(w,\phi)}$ and $w' = w$.

Proof of (2). Let $M = (W, w)$ be a k-model of T. It is easy to see that $M' = (W^{(w,\phi)}, w)$ is a model of $K\phi$. All we need to show is that $M' \in Min(Mod(K\phi), \leq_M)$. This follows from the above result we have proved. ■

Proposition 5 *Consider T and ϕ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond \neg K\phi$, then there exists a k-model $M = (W, w)$ of T such that
  (i) if $M \models K\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$,
  (ii) otherwise, $w' = w$ and $W' = W$;
(2) If $M = (W, w)$ is a k-model of T, then $M' = (W', w')$ is a k-model of $T \diamond \neg K\phi$, where
  (i) if $M \models K\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$,
  (ii) otherwise, $w' = w$ and $W' = W$.

Proof: To prove this proposition, we first prove the following result:

Consider T and ϕ where ϕ is objective. Let $M = (W, w)$ be a k-model of T,

$M \models K\phi$, and $M' = (W', w)$, where $W' = W \cup \{w^*\}$, $w^* \models \neg\phi$. Then for any k-model $M'' \in Mod(\neg K\phi)$ and $M'' \neq M'$, $M' <_M M''$.

Since $M = (W, w)$ is a k-model of T it is easy to see that for two k-models $M' = (W', w')$ and $M'' = (W'', w'')$ such that $M' \models \mu$ and $M'' \models \mu$, and $w' = w$ and $w'' \neq w$, $M' <_M M''$. So M'' can not be a k-model of $T \diamond \mu$. In other words, a k-model of $T \diamond \mu$ must have the form $M' = (W', w)$.

From Proposition 3, to prove the result, we only need to show that for any k-model $M'' = (W'', w)$ such that $M'' \in Mod(\neg K\phi)$ and $W'' \neq W \cup \{w^*\}$ where $w^* \models \neg\phi$, $M' <_M M''$.

Note $M \models K\phi$. Let $M' = (W \cup \{w^*\}, w)$, where $w^* \models \neg\phi$. We first show that for any k-model $M'' = (W'', w)$ such that $M'' \models \neg K\phi$ and W'' does not have a form of $W \cup \{w_i\}$, $M' <_M M''$.

Case 1. Suppose $W' \subset W''$. This implies that $KM'' \subseteq KM' \subseteq KM$ from Proposition 1. So $M' <_M M''$ according to Definition 2 (condition (b)).
Case 2. Suppose $W'' \subset W'$. Without loss of generality, we assume that $W'' = W \cup \{w^*\} \setminus \{w_j\}$ where $w_j \in W$. This follows that $W \not\subset W''$ and $W'' \not\subset W$. So it is the case that $KM \setminus KM'' \neq \emptyset$ and $KM'' \setminus KM \neq \emptyset$. On the other hand, we have $W \subseteq W'$, from Definition 2 (i.e. condition (a)), we have $M' <_M M''$.
Case 3. Suppose $W'' \not\subset W'$ and $W' \not\subset W''$. Without loss of generality, we can assume that $W'' = W \cup \{w^*, w_i\} \setminus \{w_j\}$, where $w_j \in W$ and $w^*, w_i \notin W$. Again, this results to the situation that $W \not\subset W''$ and $W'' \not\subset W$. From the above discussion, it implies that $M' <_M M''$.

Now we show that for any k-model M'' that is of the form $M'' = (W \cup \{w_i\}, w)$ and w_i is any world such that $w_i \models \neg\phi$ (note $M \models K\phi$), $M' \not\leq_M M''$ and $M'' \not\leq_M M'$. Suppose $M' \leq_M M''$. Since $W \subset W \cup \{w^*\}$, then according to Definition 2, condition (a) or (b) should be satisfied. As $W \subset W \cup \{w_i\}$, condition (a) can not be satisfied. So condition (b) must be satisfied. That is, for any ψ such that $M \models K\psi$ and $M' \not\models K\psi$, $M'' \models K\psi$. However, this implies that $KM \setminus KM' \subseteq KM \setminus KM''$, and also $KM \cap KM'' \subseteq KM \cap KM'$. From Proposition 1 (Results 2 and 4), it follows that $W \cup W' \subseteq W \cup W''$, that is, $W \cup \{w^*\} \subseteq W \cup \{w_i\}$. Obviously, this is not true. Similarly, we can show that $M'' \not\leq_M M'$. That means, both M' and M'' are in $Res(M, \mu)$. This completes our proof for the above result.

By using this result, we now prove statements (1) and (2).

<ins>proof of (1)</ins>. $M' = (W', w')$ is a k-model of $T \diamond \neg K\phi$ iff $M' \in Mod(T \diamond \neg K\phi)$ iff $M' \in Min(Mod(\neg K\phi), \leq_M)$ for some $M \in Mod(T)$. Now we prove that if $M \models K\phi$ then $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$ and $w' = w$; otherwise $M' = M = (W, w)$.

First, if $M \not\models K\phi$, that means $\models \neg K\phi$. In this case, according to Definition 3, $Res(M, \neg K\phi) = \{M\}$, i.e. no any change will be made. So $M' = M$. Now we consider $M \models K\phi$. We will show that in this case $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$. Assume that this is not the case. So either $w' \neq w$ or $W' \neq W \cup \{w^*\}$, where $w^* \models \neg\phi$. If $w' \neq w$, it is quite clear that for any $M'' = (W'', w) \in Mod(\neg K\phi)$, $M'' <_M M'$, this contradicts with $M' \in Min(Mod(\neg K\phi), \leq_M)$. Now assume $W' \neq W \cup \{w^*\}$, where $w^* \models \neg\phi$. In this case, from the above result we proved, it follows that there exists a k-model $M^* = (W^*, w)$, where $W^* = W \cup \{w^*\}$, where $w^* \models \neg\phi$, $M* <_M M'$. Hence it also contradicts with $M' \in Min(Mod(\neg K\phi), \leq_M)$.

Proof of (2). Let $M = (W, w)$ be a k-model of T and $M \models K\phi$. It is easy to see that $M' = (W', w)$, where $W' = W \cup \{\{w^*\}$ and $w^* \models \neg\phi$ is a k-model of $\neg K\phi$. All we need to show it aht $M' \in Min(Mod(\neg K\phi, \leq_M)$. This is followed from the above result we have proved. On the other hand, if $M \not\models K\phi$, then it is obviously $Res(M, \neg K\phi) = \{M\}$ which implies $M' = M$. ■

Proposition 6 *Consider T and $\mu \equiv K\phi \vee K\neg\phi$ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond (K\phi \vee K\neg\phi)$, then there exists a k-model $M = (W, w)$ of T such that $w = w'$ and $W' = W^{(w,\phi)}$, or $w = w'$ and $W' = W^{(w,\neg\phi)}$;

(2) If $M = (W, w)$ is a k-model of T, then $M' = (W', w')$ is a k-model of $T \diamond (K\phi \vee K\neg\phi)$, where $w' = w$ and $W' = W^{(w,\phi)}$, or $w' = w$ and $W' = W^{(w,\neg\phi)}$.

Proof: We first prove the following result:

> Consider T and $\mu \equiv K\phi \vee K\neg\phi$ where ϕ is objective. Let $M = (W, w)$ be a k-model of T, and $M' = (W^{(w,\phi)}, w)$ or $M' = (W^{(w,\neg\phi)}, w)$. Then for any $M'' \in Mod(K\phi \vee K\neg\phi)$ and $M'' \neq M'$, $M' <_M M''$.

This result can be proved in the same way as the proof of the result in the proof of Proposition 4 described earlier.

Now by using this result, we prove statements (1) and (2).

Proof of (1). $M' = (W', w')$ is a k-model of $T \diamond (K\phi \vee K\neg\phi)$ iff $M' \in Mod(T \diamond (K\phi \vee K\neg\phi))$ iff $M' \in Min(Mod(K\phi \vee K\neg\phi), \leq_M)$ for some $M \in Mod(T)$. Now we show that $W' = W^{(w,\phi)}$ and $w' = w$, or $W' = W^{(w,\neg\phi)}$ and $w' = w$. Suppose this is not the case. Then let $M^* = (W^{(w,\phi)}, w)$ or $M^* = (W^{(w,\neg\phi)}, w)$. But the above result, we have $M^* <_M M'$. This contradicts with $M' \in Min(Mod(K\phi \vee K\neg\phi), \leq_M)$. Hence our assumption is wrong and it must be the case $W' = W^{(w,\phi)}$ or $W' = W^{(w,\neg\phi)}$.

proof of (2). Let $M = (W, w)$ be a k-model of T. It is easy to see that $M' = (W^{(w,\phi)}, w)$ or $M' = (W^{(w,\neg\phi)}, w)$ is a k-model of $K\phi \vee K\neg\phi$. All we need to show is that $M' \in Min(Mod(K\phi \vee K\neg\phi), \leq_M)$. This follows from the above result we proved. ■

Proposition 7 *Consider T and $\mu \equiv \neg K\phi \wedge \neg K\neg\phi$ where ϕ is objective.*

(1) If $M' = (W', w')$ is a k-model of $T \diamond \mu$, then there exists a k-model $M = (W, w)$ of T such that
(i) if $M \models K\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$;
(ii) if $M \models K\neg\phi$, then $w' = w$ and $W' = W \cup \{w^*\}$, where $w^* \models \phi$;
(iii) otherwise, $w' = w$ and $W' = W$.

(2) If $M = (W, w)$ is a k-model of T, then $M' = (W', w)$ is a k-model of $T \diamond \mu$ where
(i) if $M \models K\phi$, then $W' = W \cup \{w^*\}$, where $w^* \models \neg\phi$;
(ii) if $M \models K\neg\phi$, then $W' = W \cup \{w^*\}$, where $w^* \models \phi$;
(iii) otherwise, $W' = W$.

Proof: We first prove the following result:

Consider T and $\mu = \neg K\phi \wedge \neg K\neg\phi$. Let $M = (W, w)$ be a k-model of T, and $M' = (W', w)$, where $W' = W \cup \{w^*\}$ and $w^* \models \neg\phi$ if $M \models K\phi$, $W' = W \cup \{w^*\}$ and $w^* \models \phi$ if $M \models K\neg\phi$, and $W' = W$ otherwise. Then for any k-model $M'' \in Mod(\neg K\phi \wedge \neg K\neg\phi)$ and $M' \neq M'$, $M' <_M M''$.

The proof for this result is similar to the proof of the result in Proposition 5. Let $\mu = \neg K\phi \wedge \neg K\neg\phi$. Firstly, it is easy to see that for any $M'' = (W'', w'') \in Mod(\neg K\phi \wedge \neg K\neg\phi)$ where $w'' \neq w$, $M' <_M M''$. So, to prove the result, we only need to show that for any k-model $M'' = (W'', w) \in Mod(\neg K\phi \wedge \neg K\neg\phi)$ and $W'' \neq W \cup \{w^*\}$, $M' <_M M''$.

Let $M' = (W \cup \{w^*\}, w)$, where $w^* \models \neg\phi$ if $M \models K\phi$ and $w^* \models \phi$ if $M \models K\neg\phi$. We first prove that for any k-model $M'' = (W'', w)$ such that $M'' \models \mu$ and W'' does not have a form of $W \cup \{w_i\}$, $M' <_M M''$.

Suppose $M \models K\phi$. Clearly $M' \models \mu$.
Case 1. Consider a k-model $M'' = (W'', w)$ where $W' = W \cup \{w^*\} \subset W''$. Note that $M'' \models \mu$ as well. However, from Proposition 1, we have $KM'' \subset KM' \subseteq KM$. So $M' <_M M''$ according to Definition 2 (i.e. condition (b)).
Case 2. Suppose $W'' \subset W'$. Without loss of generality, we assume that $W'' = W \cup \{w^*\} \setminus \{w_j\}$ where $w_j \in W$. This follows that $W \not\subset W''$ and $W'' \not\subset W$. So it is the case that $KM \setminus KM'' \neq \emptyset$ and $KM'' \setminus KM \neq \emptyset$. On the other hand, we have $W \subseteq W'$, from Definition 2 (i.e. condition (a)), we have $M' <_M M''$.
Case 3. Now suppose $W'' \not\subset W'$ and $W' \not\subset W'$. Without loss of generality, we can assume that $W'' = W \cup \{w^*, w_i\} \setminus \{w_j\}$, where $w_j \in W$ and $w^*, w_i \notin W$.

Again, this results to the situation that $W \not\subset W''$ and $W'' \not\subset W$. From the above discussion, it implies $M' <_M M''$.

Following the same way as above, we can prove that under the condition that $M \models K\neg\phi$ and $M' = (W \cup \{w^*\}, w)$ where $w^* \models \phi$, for any k-model $M'' = (W'', w)$ such that $M'' \models \mu$ and W'' does not have a form of $W \cup \{w_j\}$ $M' <_M M''$.

Now we show that for any k-model M'' that is of the form $M'' = (W \cup \{w_i\}, w)$ and w_i is any world such that $w_i \models \neg\phi$ if $M \models K\phi$ or $w_i \models \phi$ if $M \models K\neg\phi$, $M' \not\leq_M M''$ and $M'' \not\leq_M M'$. Suppose $M' \leq_M M''$. Since $W \subset W \cup \{w^*\}$, then according to Definition 2, condition (a) or (b) should be satisfied. As $W \subset W \cup \{w_i\}$, condition (a) can not be satisfied. So condition (b) must be satisfied. That is, for any ψ such that $M \models K\psi$ and $M' \not\models K\psi$, $M'' \models K\psi$. However, this implies that $KM \setminus KM' \subseteq KM \setminus KM''$, and also $KM \cap KM'' \subseteq KM \cap KM'$. From Proposition 1 (Results 2 and 4), it follows that $W \cup W' \subseteq W \cup W''$, that is, $W \cup \{w^*\} \subseteq W \cup \{w_i\}$. Obviously, this is not true. Similarly, we can show that $M'' \not\leq_M M'$. That means, both M' and M'' are in $Res(M, \mu)$. This completes our proof for this result.

By using the above result, we now prove statements (1) and (2).

Proof of (1). M' is a k-model of $T \diamond \mu$ iff $M' \in Min(Mod(\mu), \leq_M)$ for some $M \in Mod(T)$. We show that M' must be of the form as stated in statement (1). Similarly to the proof of Proposition 5 earlier, if M' is not of such form, then for a k-model M^* which has such form, we have $M^* <_M M'$ according to the above result.

Proof of (2). Let $M = (W, w) \in Mod(T)$. All we need to show is that for any M' that is of the form as stated in (2), $M' \in Min(Mod(\mu), \leq_M)$. This is indeed the case as showed by the above result. ■

Lemma 1 *Let $M = (W, w), M_1 = (W_1, w_1)$ and $M_2 = (W_2, w_2)$ be three k-models.*

(1) Deciding whether $KM \setminus KM_2 \neq \emptyset$ and $KM_2 \setminus KM \neq \emptyset$ has time complexity $\mathcal{O}(|W| \times |W_2|)$.
(2) Deciding whether $KM \setminus KM_1 \subseteq KM \setminus KM_2$ has time complexity $\mathcal{O}(|W_1| \times (|W| + |W_2|))$.
(3) Deciding whether $KM_1 \setminus KM \subseteq KM_2 \setminus KM$ has time complexity $\mathcal{O}(|W_1| \times |W| \times |W_2|)$.

Proof: Result 1 is equivalent to deciding whether $W \not\subset W_2$ and $W_2 \not\subset W$ (proper set inclusion). Obviously, this can be verified in $\mathcal{O}(|W| \times |W_2|)$ time.

Now we prove Result 2. From set inclusion and intersection properties, it is easy to see that $KM \backslash KM_1 \subseteq KM \backslash KM_2$ iff $KM_2 \cap KM \subseteq KM_1 \cap KM$. Then from Proposition 1 (Results 2 and 4), it follows that $KM_2 \cap KM \subseteq KM_1 \cap KM$ iff $W_1 \cup W \subseteq W_2 \cup W$. Obviously, checking whether $W_1 \cup W \subseteq W_2 \cup W$ can be done in time $\mathcal{O}(|W_1| \times (|W| + |W_2|))$.

Finally we prove Result 3. First it is easy to show that $KM_1 \setminus KM \subseteq KM_2 \setminus KM$ iff $KM_1 \cup KM \subseteq KM_2 \cup KM$ iff $KM_1 \subseteq KM_2 \cup KM$. We will now show that

> $KM_1 \not\subseteq KM_2 \cup KM$ if and only if there exists $w \in W$ and $w_2 \in W_2$, such that $w, w_2 \notin W_1$.

($\Rightarrow$) Let $\phi \in KM_1$ and $\phi \notin KM_2 \cup KM$. This implies $\phi \notin KM_2$ and $\phi \notin KM$. This implies there exists $w_2 \in W_2$ and $w \in W$ such that $w_2 \not\models \phi$ and $w \not\models \phi$. These w and w_2 are both not in W_1 as ϕ is true in all worlds of W_1.

($\Leftarrow$) Let $\phi = \neg(\bigwedge w) \wedge \neg(\bigwedge w_2)$.[5] It is easy to see that $\phi \notin KM_2$ and $\phi \notin KM$, and ϕ holds in all worlds in W_1. Hence, $KM_1 \not\subseteq KM_2 \cup KM$.

Now to determine if there exists $w \in W$ and $w_2 \in W_2$, such that $w, w_2 \notin W_1$, we need to go through each worlds in W and W_2 and check if they are in W_1 or not. All these checks can be done in time $\mathcal{O}(|W_1| \times |W| \times |W_2|)$. ∎

To prove Theorem 5, we need to prove the following lemma first.

Lemma 4 *Let $X, Y, \hat{X}, \hat{Y}$ be sets of propositional atoms and a be a propositional atom, where $|X| = |\hat{X}|$, $|Y| = |\hat{Y}|$ and any two sets of $X, Y, \hat{X}, \hat{Y}$ and $\{a\}$ are disjoint. Suppose ϕ is an objective formula only using letters from set $X \cup Y$. Let T and μ be the following two S5 formulas respectively:*

$$
\begin{aligned}
&T = \gamma_1 \vee \gamma_2, \textit{ where}\\
&\gamma_1 = (((X \equiv \hat{X}) \equiv \phi) \equiv a) \wedge (Y \wedge \neg\hat{Y}) \wedge\\
&\qquad \neg K \neg(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}),\\
&\gamma_2 = (((X \wedge \neg\hat{X}) \equiv \neg\phi) \equiv \neg a) \wedge \hat{Y}, \textit{ and}\\
&\mu = K(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}) \vee\\
&\qquad K(\neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}) \vee\\
&\qquad K(\neg a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y})\,^{6}.
\end{aligned}
$$

Then $T \diamond \mu$ is knowledge gradual.

---

[5] Note that we use notion $\bigwedge w$ to denote the conjunction of all propositional atoms that occur in w. If an atom is not in w, its negation will be in $\bigwedge w$. For instance, if $w = \{a, c\}$, then $\bigwedge w = a \wedge \neg b \wedge c$ when a, b and c are the only propositional atoms in the language.

[6] Recall that $\bigvee \neg Y = \bigvee_{y_i \in Y} \neg y_i$.

Proof: To prove $T \diamond \mu$ to be knowledge gradual, we need to show that for any k-model $M = (W, w) \in Mod(T)$, if $M' = (W', w') \in Res(M, \mu)$, then either $W \subseteq W'$ or $W' \subseteq W$. From the construction of T, it is easy to see that if $M \in Mod(T)$, then either $M \models \gamma_1$ or $M \models \gamma_2$, but $M \not\models \gamma_1 \wedge \gamma_2$. Based on this observation, our proof consists of two cases.

Case 1. Let $M = (W, w) \in Mod(\gamma_1)$. Since $M \models \gamma_1$, we have

$$w = X_1 \cup \hat{X}_1 \cup Y \cup \{a\}, \text{ where } X_1 \cup Y \models \phi \text{ for some } X_1 \subseteq X.$$

Furthermore, since $M \models \neg K \neg (a \wedge X \wedge \neg \hat{X} \wedge Y \wedge \hat{Y})$, there exists a world $w^* \in W$ such that

$$w^* = X \cup Y \cup \hat{Y} \cup \{a\}.$$

Now we specify a k-model of μ as follows:

$$M^* = (\{w^*\}, w^*).$$

It is easy to see that $M^* \models K(a \wedge X \wedge \neg \hat{X} \wedge Y \wedge \hat{Y})$. So $M^* \models \mu$. Furthermore, M^* is the unique k-model of $K(a \wedge X \wedge \neg \hat{X} \wedge Y \wedge \hat{Y})$. We prove M^* is the unique k-model in $Res(M, \mu)$.

Note $Diff(w, w^*) = (X \setminus X_1) \cup \hat{X}_1 \cup \hat{Y}$. Besides M^*, μ has two other types of k-models:

$$M_1 = (W_1, w_1), \text{ where } w_1 = X \cup Y_1 \cup \hat{Y}, \text{ where } Y_1 \subset Y \ (Y_1 \neq Y), \text{ and}$$
$$M_2 = (W_2, w_2), \text{ where } w_2 = X \cup Y \cup \hat{Y}.$$

Note that

$$M_1 \models K(\neg a \wedge X \wedge \neg \hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}), \text{ and}$$
$$M_2 \models K(\neg a \wedge X \wedge \neg \hat{X} \wedge Y \wedge \hat{Y}).$$

Consider

$$Diff(w, w_1) = (X \setminus X_1) \cup \hat{X}_1 \cup (Y \setminus Y_1) \cup \hat{Y} \cup \{a\},$$
$$Diff(w, w_2) = (X \setminus X_1) \cup \hat{X}_1 \cup \hat{Y} \cup \{a\}.$$

Clearly, we have

$$Diff(w, w^*) \subset Diff(w, w_1), \text{ and}$$
$$Diff(w, w^*) \subset Diff(w, w_1).$$

So $Res(M, \mu) = \{M^*\}$. Also observe that $M^* = (\{w^*\}, w^*)$, $\{w^*\} \subset W$.

Case 2. Let $M = (W, w) \in Mod(\gamma_2)$. We have

$w = X \cup Y_1 \cup \hat{Y}$, where $X \cup Y_1 \models \neg\phi$ for some $Y_1 \subseteq Y$.

If $Y_1 \neq Y$, then we have

$$w \models \neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}),$$

this implies that there exists a subset W_1 of W where for each $w_i \in W_1$,

$$w_i \models (\neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}).$$

By specifying W_1 to be the maximal such subset of W, it is easy to note that $M_1 = (W_1, w)$ is a k-model in $Res(M, \mu)$.

On the other hand, if $Y_1 = Y$, then we have

$$w \models (\neg a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}),$$

this implies that there exists a subset W_2 of W where for each $w_i \in W_2$,

$$w_i \models (\neg a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}).$$

Similarly, by specifying W_2 to be the maximal such subset of W, it is easy to note that $M_2 = (W_2, w)$ is a k-model in $Res(M, \mu)$.

Since in both cases, we have $W_1 \subseteq W$ and $W_2 \subseteq W$, this follows that for any k-model M of T where $M \models \gamma_2$, every resulting k-model after updating M with μ only increases the knowledge from M. ■

Theorem 5 *Model checking for knowledge update is Σ_2^P-complete. The hardness holds even if the update is knowledge gradual.*

Proof: According to Theorem 4, we only need to prove the hardness part. This part is based on a variation of the proof of Lemma 4. We prove the hardness by giving a polynomial transformation from deciding the validity of $\exists X \forall Y E$, where E is a Boolean expression using propositional atoms over $X \cup Y$. We construct T, μ and a k-model M^* over propositional atoms $X \cup Y \cup \hat{X} \cup \hat{Y} \cup \{a\}$, where $|\hat{X}| = |X|$ and $|\hat{Y}| = |Y|$, and any two sets among $X, Y, \hat{X}, \hat{Y}$ and $\{a\}$ are disjoint.

$$\begin{aligned}
&T = \gamma_1 \vee \gamma_2, \text{ where} \\
&\gamma_1 = (((X \equiv \hat{X}) \equiv E) \equiv a) \wedge (Y \wedge \neg\hat{Y}) \wedge \\
&\qquad \neg K \neg(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}), \\
&\gamma_2 = (((X \wedge \neg\hat{X}) \equiv \neg E) \equiv \neg a) \wedge \hat{Y}, \\
&\mu = K(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}) \vee \\
&\qquad K(\neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}) \vee \\
&\qquad K(\neg a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}),
\end{aligned}$$

$M^* = (W^*, w^*)$, where
$W^* = \{w^*\}$, $w^* = X \cup Y \cup \hat{Y} \cup \{a\}$.

Note that

$$M^* \models K(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}).$$

So M^* is a k-model of μ. Furthermore, it is the unique k-model of $K(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y})$. From Lemma 10, we know that $T \diamond \mu$ is knowledge gradual. Now we will show that M^* is a k-model of $T \diamond \mu$ if and only if $\exists X \forall Y E$ is valid.

($\Rightarrow$) Suppose $\exists X \forall Y E$ is valid. Then for some $X_1 \subseteq X$, $X_1 \cup Y \models E$. We specify a k-model of γ_1 as follows:

$$M = (W, w), \text{ where } w = X_1 \cup \hat{X}_1 \cup Y \cup \{a\}.$$

Since $M \models \neg K \neg(a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y})$, it is clear that the world w^* must be in W, i.e. $w^* \in W$. With the same justification as described in the proof of Lemma 4, we conclude that M^* is a k-model of updating M with μ.

($\Leftarrow$) Suppose $\exists X \forall Y E$ is not valid. That is, $\forall X \exists Y \neg E$ is valid. Then $X \cup Y_1 \models \neg E$ for some $Y_1 \subseteq Y$. In this case, T has the following type of k-models:

$$M = (W, w), \text{ where } w = X \cup Y_1 \cup \hat{Y}.$$

Note that $w \models ((X \wedge \neg\hat{X} \equiv \neg E \equiv \neg a) \wedge \hat{Y})$. That is, M is a k-model of γ_2.

If $Y_1 \neq Y$, then we have

$$w \models (\neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}).$$

We now specify a k-model of μ as follows: $M_1 = (W_1, w_1)$, where $w_1 = w$ and W_1 is the maximal subset of W such that for each $w_i \in W_1$,

$$w_i \models (\neg a \wedge X \wedge \neg\hat{X} \wedge (\bigvee \neg Y) \wedge \hat{Y}).$$

Since

$$Diff(w, w^*) = (X \setminus X_1) \cup \hat{X}_1 \cup \hat{Y}, \text{ and}$$
$$Diff(w, w_1) = \emptyset \subset Diff(w, w^*),$$

M^* is not a k-model in $Res(M, \mu)$.

If $Y_1 = Y$, then we have

$$w \models (\neg a \wedge X \wedge \neg\hat{X} \wedge Y \wedge \hat{Y}).$$

Again, we can specify a k-model of μ as follows: $M_2 = (W_2, w_2)$, where $w_2 = w$

and W_2 is maximal subset of W such that for each $w_i \in W_2$,

$$w_i \models (\neg a \wedge X \wedge \neg \hat{X} \wedge Y \wedge \hat{Y}).$$

Since $Diff(w, w_2) = \emptyset \subset Diff(w, w^*)$, M^* is not a k-model in $Res(M, \mu)$ in this case either.

Finally, suppose for some $X_1 \subseteq X$ and $Y_1 \subseteq Y$, E is evaluated to be true on $X_1 \cup Y_1$, i.e. $X_1 \cup Y_1 \models E$. Without loss of generality, we can assume $Y_1 \neq Y$ [7]. This implies that γ_1 does not have a k-model under this situation. Therefore, if $\exists X \forall Y E$ is not valid, all k-models of T must be k-models of γ_2. ■

Theorem 8 *Model checking for knowledge increased update is NP-complete.*

Proof: Membership proof. Given T, μ and $M' = (W', w')$. To deciding whether $M' \in Mod(T \diamond \mu)$, we only need to show that for some $M \in Mod(T)$, $M' \in Res(M, \mu)$. A guess of $M = (W, w)$ and verifying $M \models T$ can be done in polynomial time. Since $T \diamond \mu$ is knowledge increased, to decide $M' \in Res(M, \mu)$, we only need to check: (1) $w = w'$, and (2) for any $w^* \in W$, $w^* \in W'$ iff $w^* \models \phi^\mu$ or $w^* \models \neg \phi^\mu$ for some $\phi^\mu \in Sub^o(\mu)$. Obviously, both (1) and (2) can be checked in polynomial time. So the problem is in NP.

Hardness proof. The hardness is proved by transforming the NP-complete SAT problem to a gaining knowledge update that has been showed to be knowledge increased. Let E be a CNF on the set of propositional atoms X. We construct formulas T, μ and a k-model M' over two disjoint sets X and $\hat{X}$ where $|X| = |\hat{X}|$.

$T = (X \equiv \hat{X}) \wedge \neg K(X \equiv \hat{X})$,
$\mu \equiv K(X \equiv \hat{X} \vee \neg E)$, and
$M' = (W', w')$, where
$W' = \{w'\}$, $w' = X \cup \hat{X}$.

Clearly, $M \models \mu$. We will show that E is satisfiable iff $M' \in Mod(T \diamond \mu)$. Note that since $T \models X \equiv \hat{X} \vee \neg E$ and $\mu = K(X \equiv \hat{X} \vee \neg E)$, $T \diamond \mu$ is a gaining knowledge update that is knowledge increased according to Theorem 5.

($\Rightarrow$) Suppose E is satisfiable. Let $X_1 \subseteq X$ such that $X_1 \models E$. We specify a k-model as follows:

$$M^* = (W^*, w^*),$$

[7] Note that this assumption is always feasible. For instance, if $X_1 \cup Y \models E$, we can expand Y to be Y' by adding a new atom y' into Y to make $Y \neq Y'$, i.e. $Y' = Y \cup \{y'\}$, and modify E to be $E' = E \wedge \neg y'$ such that $X_1 \cup Y \models E'$ but $X_1 \cup Y' \not\models E'$.

$W^* = \{w^*, w''\}$,
$w^* = w' = X \cup \hat{X}$, and
$w'' = X_1 \cup \hat{X}_1$, where $\hat{X}_1 = \{\hat{x}_i \mid \hat{x}_i \in \hat{X}$ and $x_i \notin X_1\}$.

Since $w'' \not\models X \equiv \hat{X}$, it is easy to see that $M^* \models \neg K(X \equiv \hat{X})$. Therefore, M^* is a k-model of T. On the other hand, since $w'' \models E$ and $w'' \not\models X \equiv \hat{X}$, it follows that $W' = \{w'\} = \{w^*\} = W^{*(w^*,\phi)}$, where $\phi = (X \equiv \hat{X}) \vee \neg E$. From Lemma 6, $M' \in Res(M^*, \mu)$, so $M' \in Mod(T \diamond \mu)$.

($\Leftarrow$) Now suppose E is not satisfiable. That is, for any $X_1 \subseteq X$, $X_1 \models \neg E$. Then from Lemma 6, for any k-model of T of the form $M = (W, w)$, where $w \neq w'$, $M' \notin Res(M, \mu)$. We consider k-models of T of the form $M = (W, w)$ where $w = w'$ (note $w' \in W$). Without loss of generality, we assume that there is one world $w^* \in W$ such that $w^* \not\models X \equiv \hat{X}$, otherwise $M \models K(X \equiv \hat{X})$ and M cannot be a k-model of T. On the other hand, since E is not satisfiable, $\neg E$ must be true in each world in W. So $M \models K\neg E$ and hence $M \models K(X \equiv \hat{X} \vee \neg E)$. This implies that $W' \neq W^{(w,\phi)}$, where $\phi = (X \equiv \hat{X}) \vee \neg E$. So M' is not a k-model of $T \diamond \mu$. ■