

# Constructive Circumscription

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## Abstract

Circumscription is one of the most powerful nonmonotonic reasoning formalisms. Its classical model-theoretic semantics serves as an effective mechanism to ensure minimality in commonsense reasoning. However, as the lack of proof-theoretic feature, it is incapable of expressing derivation order that sometimes plays a crucial role in various reasoning tasks. On the other hand, the general theory of stable models has a similar logic formulation to circumscription but takes supportedness into account, although it does not satisfy the minimality criterion in general. In this paper, we introduce the notion of *constructive circumscription*. Unlike the original circumscription, constructive circumscription enforces both minimality and derivation order. Such derivation order is achieved by strengthening the underlying circumscriptive theory with a first-order formula called *explicit construction formula*. With constructive circumscription, minimal models are only considered under an explicit constructibility. We study properties of constructive circumscription in details, and compare it with circumscription, the general theory of stable models, and the FLP semantics.

## 1 Introduction

Circumscription (Lifschitz 1994; McCarthy 1980) is one of the most powerful nonmonotonic reasoning formalisms that can handle incomplete and negative information. Among various reasoning features, one major advantage of circumscription over other approaches is its classical model-theoretic semantics serving as an effective mechanism to ensure minimality in commonsense reasoning. Circumscription at its core is a translation of a first-order theory into a stronger second-order theory that circumscribes the models of the first-order theory to be “minimal” with respect to a set of predicates. This minimality requirement models the commonsense reasoning that things are to be expected unless otherwise specified. Although powerful and with a good grounding on the well-developed system of classical logic, circumscription has one inherent restriction: it is incapable of expressing proof-theoretic concepts, and thus, is sometimes not suitable for reasoning tasks explicitly associated with an embedded derivation path

In recent years, the *general theory of stable models* (Ferraris et al. 2011) has emerged as an important nonmonotonic formalism for arbitrary first-order theories. The stable model semantics, originally proposed in (Gelfond and Lifschitz 1988), has its root in *logic programming* and the semantics of *negation as failure* (Clark 1987). Compared to circumscription, the original intended meaning of logic programming is not defined through minimality, but through *supportedness*, in the sense that a fact is derived to be true only if it is supported by other facts that are already true under negation as failure. It is a fact that the stable model semantics for traditional normal and disjunctive logic programs enjoys the desirable *anti-chain property* (minimality) (Ferraris and Lifschitz 2011), i.e., for any given program, its one stable model cannot be a strict subset of another

stable model. But, the generalization of stable models to arbitrary theories and other constructs (such as aggregates) has moved away from this property. For instance, formula (5) of Proposition 2 from (Ferraris and Lifschitz 2011):  $P(a) \wedge (Q(a) \leftrightarrow P(b)) \wedge \exists x(P(x) \rightarrow Q(x))$ , has two Herbrand stable models  $S_1 = \{p(a)\}$  and  $S_2 = \{p(a), p(b), q(a)\}$ , where  $S_1$  is a strict subset of  $S_2$ , which obviously violate the anti-chain property. The importance of the anti-chain property in commonsense reasoning has been specifically addressed and justified in significant publications (Gelfond and Lifschitz 1988; Faber et al. 2011; Ferraris and Lifschitz 2011; Truszczyński 2010).

The FLP semantics (standing for Faber, Leone, and Pfeifer) was initially proposed for the purpose of defining a semantics for disjunctive programs with arbitrary aggregates that satisfies the anti-chain property (Faber et al. 2011). Recently, this semantics has been further extended to logic programs with first-order formulas (Bartholomew et al. 2011). Even though FLP satisfies both the anti-chain and supportedness properties, it was recently observed by Shen and Wang that the FLP semantics suffers from the so-called *circular justification problem* (Shen and Wang 2012). That is, under certain circumstances, a program may have a stable model containing a fact that is solely justified by itself.<sup>1</sup>

In this paper, we introduce the notion of *constructive circumscription*. Unlike the original form of circumscription, constructive circumscription enforces both derivation order and minimality. Such derivation order is achieved by strengthening the underlying circumscriptive theory with a first-order formula so-called *explicit construction formula*. Also unlike the first-order stable model operator in (Ferraris et al. 2011) or the first-order FLP and FLPT operators in (Bartholomew et al. 2011), constructive circumscription is a form of circumscription, in the sense that it can simply be expressed as the minimal models of some first-order theory without any further assertions. We compare constructive circumscription with traditional circumscription, the general theory of stable models, and the FLP semantics, and show that constructive circumscription satisfies all the three desirable reasoning properties, and hence, provides some new features for first-order nonmonotonic reasoning.

## 2 Preliminaries

### 2.1 Logical Preliminaries

We assume a general familiarity with both first-order (FO) and *second-order* (SO) logic. We consider signature of the form  $\tau = \mathcal{C} \cup \mathcal{P} \cup \mathcal{F}$  such that  $\mathcal{C}$ ,  $\mathcal{P}$ , and  $\mathcal{F}$  are its sets of constants, predicates, and function symbols, respectively. For clarity, we refer to a signature of a formula  $\varphi$  by  $\tau(\varphi)$ . By  $\mathcal{P}_{ext}$  and  $\mathcal{P}_{int}$ , we denote the subsets of predicate symbols  $\tau(\varphi)$  called the *extensional* and *intensional* predicates, respectively. Let  $\varphi$  be an FO or SO formula, we use  $\text{Mod}(\varphi)$  to denote the set of all models of  $\varphi$ . In the spirit of *Answer Set Programming* (ASP), the extensional predicates are viewed as the input knowledge database and the intensional as the “intended” extension of the input database. Following (Ferraris et al. 2011), this extends the notion of extensional and intensional predicates to arbitrary FO theories.

In this paper, our formalization will rely on the notions of subformulas and maximal formulas.

<sup>1</sup> It is worth mentioning that although the recent work proposed by Shen and Wang ((2012)) has overcome the non-circular justification issue in the FLP semantics, their formalism, however, is rather restricted in general because not all non-circular derivation programs can be correctly captured by their approach, e.g., for the program  $\{\rightarrow p \vee q, p \rightarrow q, q \rightarrow p\}$ , it is observed that  $\{p, q\}$  cannot be a stable model in terms of Shen and Wang’s although clearly, neither  $p$  nor  $q$  complies with a circular derivation in this program due to the rule  $\rightarrow p \vee q$ .

Let  $\varphi$  be an FO formula. A *subformula* of  $\varphi$  is a well-formed formula that has at least one occurrence in  $\varphi$ . We use  $\text{SUBF}[\varphi]$  to denote the set of all subformulas of  $\varphi$ .

We further assume that each subformula is associated with a unique integer that would enable us to differentiate between the different occurrences of a subformula within the whole formula. Note that under this notion, elements of  $\text{SUBF}[\varphi]$  should be of the form  $(\psi, N)$  for some integer  $N$ . We omit this for simplicity and assume that it is clear from the context.

*Definition 1 (Nearest maximal formula)*

For an FO formula  $\varphi$  and  $\psi \in \text{SUBF}[\varphi]$ , if  $\psi \neq \varphi$  then the *nearest maximal formula* (NMF) of  $\psi$  under  $\varphi$ , denoted as  $\varphi \uparrow \psi$  (or just  $\uparrow \psi$  when clear from the context), is the minimal subformula of  $\varphi$  that properly contains  $\psi$ ; otherwise  $\uparrow \psi = \varphi$  if  $\psi = \varphi$ .  $\square$

*Example 1*

Assume  $\varphi = \exists y \forall x (P(x, y) \rightarrow \exists z Q(x, z))$  and let  $\psi$  be its atomic subformula “ $Q(x, z)$ ”. Then we get the following sequence of NMF:

1.  $\uparrow Q(x, z) = \exists z Q(x, z)$ ;
2.  $\uparrow \exists z Q(x, z) = (P(x, y) \rightarrow \exists z Q(x, z))$ ;
3.  $\uparrow (P(x, y) \rightarrow \exists z Q(x, z))$   
 $= \forall x (P(x, y) \rightarrow \exists z Q(x, z))$ ;
4.  $\uparrow \forall x (P(x, y) \rightarrow \exists z Q(x, z))$   
 $= \exists y \forall x (P(x, y) \rightarrow \exists z Q(x, z)) = \varphi$ .  $\square$

*Definition 2 (Iterative maximal formulas)*

Let  $\varphi$  be a formula and  $\psi \in \text{SUBF}[\varphi]$ . Then the iterative maximal formulas of  $\psi$  under  $\varphi$ , denoted as  $\uparrow_k \psi$ , is defined inductively as follows:

1.  $\uparrow_0 \psi = \psi$ ;
2.  $\uparrow_{k+1} \psi = \uparrow(\uparrow_k \psi)$  if  $\uparrow_k \psi \neq \varphi$ , otherwise  $\uparrow_{k+1} \psi = \uparrow_k \psi$  (i.e., if  $\uparrow_k \psi = \varphi$ ).  $\square$

Note that for any  $\varphi$  and  $\psi$ , we have  $\uparrow_\infty \psi = \varphi$ , which denotes the fixpoint of  $\uparrow_k \psi$ . In Example 1, we have  $\uparrow_\infty Q(x, z) = \uparrow_3 Q(x, z) = \varphi$ .

## 2.2 Circumscription, Stable Models and FLP: Overview

Let  $\varphi$  be an FO sentence with a tuple  $\mathbf{P} = P_1 \dots P_n$  of intensional predicate symbols occurring in  $\varphi$  (note that  $\varphi$  may also contain extensional predicate symbols). Assume  $\mathbf{U} = U_1 \dots U_n$  where  $U_1, \dots, U_n$  are fresh predicate symbols not mentioned in  $\varphi$  and each  $U_i$  has the same arity of  $P_i$  for  $1 \leq i \leq n$ . By  $\text{SM}_{\mathbf{P}}[\varphi]$ , we denote the SO formula

$$\varphi \wedge \neg \exists \mathbf{U} [\mathbf{U} < \mathbf{P} \wedge \varphi^*(\mathbf{U})], \quad (1)$$

where  $\mathbf{U} < \mathbf{P}$  is the shorthand for the formula

$$\bigwedge_{1 \leq i \leq n} \forall \mathbf{x} (U_i(\mathbf{x}) \rightarrow P_i(\mathbf{x})) \wedge \neg \bigwedge_{1 \leq i \leq n} \forall \mathbf{x} (P_i(\mathbf{x}) \rightarrow U_i(\mathbf{x})),$$

and  $\varphi^*(\mathbf{U})$  is recursively defined as follow:

1.  $\perp^* = \perp$ ;
2.  $P_i(\mathbf{x})^* = U_i(\mathbf{x})$  (for  $1 \leq i \leq n$ );

3.  $(\odot x\varphi)^* = \odot x\varphi^*$  (for  $\odot \in \{\exists, \forall\}$ );
4.  $(\varphi \odot \psi)^* = \varphi^* \odot \psi^*$  (for  $\odot \in \{\vee, \wedge\}$ );
5.  $(\varphi \rightarrow \psi)^* = (\varphi^* \rightarrow \psi^*) \wedge (\varphi \rightarrow \psi)$ ,

and where  $\neg\varphi$  is viewed as  $\varphi \rightarrow \perp$ . Then we have that a  $\tau(\varphi)$ -structure  $\mathcal{M}$  is a **P-stable model** of  $\varphi$  (or simply called a *stable model* if there is no confusion) iff  $\mathcal{M} \models \text{SM}_{\mathbf{P}}[\varphi]$  (Ferraris et al. 2011).

If we simply define the case for  $(\varphi \rightarrow \psi)^*$  as  $(\varphi^* \rightarrow \psi^*)$  (i.e., omitted the  $(\varphi \rightarrow \psi)$  part), then  $\text{SM}_{\mathbf{P}}[\varphi]$  becomes  $\text{CIRC}_{\mathbf{P}}[\varphi]$ , where **CIRC** is the circumscription operator as originally defined by McCarthy (1980) and further developed by Lifschitz (1994).

Recently, Bartholomew et al. (2011) proposed an FO counterpart of the FLP semantics, as presented in (Faber et al. 2011), via the so-called “modified circumscription”<sup>2</sup>. This FO counterpart of FLP is only applicable to a certain class of FO theories called “general logic programs”. A *general logic program* is a set of FO formulas called *rules* of the form  $B \rightarrow H$ , where  $B$  and  $H$  are arbitrary FO formulas called the rule’s *body* and *head*, respectively. Similarly to the FO stable model, the FO FLP is also defined via “FO reducts.” Indeed, given a general logic program  $\Pi$ , the FO FLP semantics is defined via the SO sentence  $\text{FLP}_{\mathbf{P}}[\widehat{\Pi}]$  as follows:

$$\widehat{\Pi} \wedge \neg\exists\mathbf{U}[\mathbf{U} < \mathbf{P} \wedge \widehat{\Pi}^{\Delta}(\mathbf{U})], \quad (2)$$

where:  $\widehat{\Pi}$  is the conjunctions of all the universal closures of the rules  $B \rightarrow H \in \Pi$ ;  $\widehat{\Pi}^{\Delta}(\mathbf{U})$  denotes all the universal closures of the formulas of the form  $B \wedge B(\mathbf{U}) \rightarrow H(\mathbf{U})$ ; and  $B(\mathbf{U})$  (similarly for  $H(\mathbf{U})$ ) denotes the formula obtained from  $B$  by replacing all the occurrences of predicates from  $\mathbf{P}$  by those corresponding ones from  $\mathbf{U}$ . Then we say that a  $\tau(\Pi)$ -structure  $\mathcal{M}$  is an *FLP stable model* of  $\Pi$  iff  $\mathcal{M} \models \text{FLP}_{\mathbf{P}}[\widehat{\Pi}]$ .

### 3 Constructive Circumscription

The basic idea of constructive circumscription is to circumscribe only those extents of predicates that can be “explicitly constructed” or justified within the underlying formula. As an example, let  $\varphi$  be the following sentence:

$$\forall x(R(x) \rightarrow Q(x)) \quad (3)$$

$$\wedge \forall x(S(x) \rightarrow R(x) \vee P(x)) \quad (4)$$

$$\wedge \forall x\neg\neg P(x) \quad (5)$$

$$\wedge \forall x\neg\neg Q(x), \quad (6)$$

where  $S$  is the only extensional predicate. Then since  $\forall x\neg\neg P(x)$  and  $\forall x\neg\neg Q(x)$  are classically equivalent to  $\forall xP(x)$  and  $\forall xQ(x)$ , respectively,  $\text{CIRC}_{PQR}[\varphi]$  will entail both  $\forall xP(x)$  and  $\forall xQ(x)$ . However, if “ $\neg$ ” is explained as negation as failure, the intended extents of  $PQR$  should only be those ones of  $P$  that are contained in  $S$  (via (4)), and those ones of  $Q$  that are contained in  $R$  (via (3)), while  $R$  in turn is contained in  $S$  (via (4)). In other words, (5) and (6) here are treated as constraints when negation as failure is considered. Thus, in contrast with traditional circumscription, constructive circumscription aims to circumscribe subject to the intended extents of  $P$  and  $Q$ .

<sup>2</sup> Note that the SM operator is also viewed as a form of modified circumscription.

Before we formally introduce the notion of constructive circumscription, we will need some useful notions from (Ferraris et al. 2011). We say that an atom  $P(\mathbf{x})$  has a *positive* occurrence in a formula  $\varphi$  if it is within a scope of an antecedent of an *even* number of implications. If the number of such implications are *odd*, then we say it has a *negative* occurrence. In addition, if the number of such implications are zero, then we say it is *strictly positive*.

### 3.1 Atomic Construction Formulas

An *atomic construction formula* encodes how a particular occurrence of an atom might explain (or construct) the inclusion of a predicate's extent in a model. As a motivating example, consider the following sentence:

$$\forall y(S(y) \rightarrow \exists x(Q(x, y) \rightarrow P(x))).$$

Then an explanation for some extent, for instance,  $P(a)$  of  $P$ , is that: (1) there exists some  $x$  such that  $P(x)$  holds with  $x = a$ ; (2)  $Q(x, y)$  also holds for some  $y$  given  $x = a$ ; and (3)  $S(y)$  also holds. That is, we backtrack the derivation along the chain of implications that lead to the atom  $P(x)$ .

For a pair of predicates  $(P, Q)$ , we introduce a new predicate  $\leq_{PQ}$ , called the *comparison predicate*, whose arity is the sum of the arities of  $P$  and  $Q$ . Then for two atoms  $P(\mathbf{x})$  and  $Q(\mathbf{y})$ , by  $\widehat{P(\mathbf{x})} < \widehat{Q(\mathbf{y})}$ , we denote the conjunction  $\leq_{QP}(\mathbf{x}, \mathbf{y}) \wedge \neg \leq_{PQ}(\mathbf{y}, \mathbf{x})$ , such that its intuitive meaning is that  $P(\mathbf{x})$  is used for deriving  $Q(\mathbf{y})$  and not the other way around.

Now for a given formula  $\varphi$  and atom  $P(\mathbf{x})$ , let  $\varphi^{<P(\mathbf{x})}$  denote the formula obtained from  $\varphi$  defined inductively as follows (where in the following,  $\odot \in \{\exists, \forall\}$  and  $\oplus \in \{\rightarrow, \wedge, \vee\}$ ):

1.  $(\odot \mathbf{y} \varphi)^{<P(\mathbf{x})} = \odot \mathbf{y}(\varphi^{<P(\mathbf{x})})$  (move into quantifiers);
2.  $(\varphi \oplus \psi)^{<P(\mathbf{x})} = (\varphi^{<P(\mathbf{x})} \oplus \psi^{<P(\mathbf{x})})$  (commute on connectives);
3.  $Q(\mathbf{y})^{<P(\mathbf{x})} = \widehat{Q(\mathbf{y})} < \widehat{P(\mathbf{x})}$  if  $Q(\mathbf{y})$  is a positive occurrence in  $\varphi$ , and  $Q(\mathbf{y})^{<P(\mathbf{x})} = Q(\mathbf{y})$  otherwise;
4.  $\perp^{<P(\mathbf{y})} = \perp$ ,

Intuitively,  $\varphi^{<P(\mathbf{x})}$  encodes how  $\varphi$  can be established (derived) without using  $P(\mathbf{x})$  by also enforcing  $\widehat{Q(\mathbf{y})} < \widehat{P(\mathbf{x})}$  for each positively occurring atom  $Q(\mathbf{y})$  of  $\varphi$ .

#### Definition 3 (Atomic construction formula)

For a formula  $\varphi$ , let  $P(\mathbf{x}) \in \text{SUBF}[\varphi]$  and  $\mathbf{y}$  a fresh tuple of distinct variables of arity  $\mathbf{x}$ . We define the *atomic construction formula* of  $P(\mathbf{x})$  under  $\varphi$ , denoted as  $\uparrow_k P(\mathbf{x})[\mathbf{y}]$ <sup>3</sup>, inductively as follows:

1.  $\uparrow_0 P(\mathbf{x})[\mathbf{y}]$  is defined as  $\mathbf{y} = \mathbf{x}$ .
2. assuming  $\uparrow_k P(\mathbf{x})[\mathbf{y}] = \phi$  and  $\uparrow_k P(\mathbf{x}) = \psi$ , we define  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}]$  based on

<sup>3</sup> Note that we replaced ' $\uparrow$ ' by ' $\uparrow$ ' to differentiate the two different kinds.

the following cases about  $\uparrow_{k+1} P(\mathbf{x})$ , such that

$$\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}] = \begin{cases} \exists x\phi, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = \exists x\psi \\ \forall x\psi \wedge \exists x\phi, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = \forall x\psi \\ \phi, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = (\xi \vee \psi) \\ \xi \wedge \phi, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = (\xi \wedge \psi) \\ \perp, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = (\psi \rightarrow \xi) \\ \phi \wedge \xi^{<P(\mathbf{y})}, & \text{if } \uparrow_{k+1} P(\mathbf{x}) = (\xi \rightarrow \psi), \end{cases}$$

where we view  $\neg\varphi$  as  $\varphi \rightarrow \perp$ .

For the case where  $\uparrow_{k+1} P(\mathbf{x}) = \uparrow_k P(\mathbf{x})$  (i.e., the fixpoint of  $\uparrow_k P(\mathbf{x})$ ), we define  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}]$  also as  $\uparrow_k P(\mathbf{x})[\mathbf{y}]$  (i.e., its fixpoint), denoted as  $\uparrow_\infty P(\mathbf{x})[\mathbf{y}]$ .  $\square$

Definition 3 defines the expression of an *explicit construction* of an extent  $P(\mathbf{a})$  of  $P$  via the occurrence of an atom  $P(\mathbf{x})$  in  $\varphi$ . That expression of the “explicit construction” via  $P(\mathbf{x})$  will eventually be the fixpoint of  $\uparrow_\infty P(\mathbf{x})[\mathbf{y}]$ . Also note that since the length of formula  $\varphi$  is finite, so is that of  $\uparrow_\infty P(\mathbf{x})[\mathbf{y}]$ .

In a nutshell, we achieve this by looking at an occurrence of an atom  $P(\mathbf{x})$  in  $\varphi$ , and then working our way “upwards” (*backtracking*), or *bottom-up* through the formula, starting from the atom  $P(\mathbf{x})$ . Indeed, for the base case, we have that  $\uparrow_0 P(\mathbf{x})[\mathbf{y}]$  is just  $\mathbf{y} = \mathbf{x}$ . Then, assuming we already have  $\uparrow_k P(\mathbf{x})[\mathbf{y}] = \phi$  and  $\uparrow_k P(\mathbf{x}) = \psi$ , we now define  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}]$  based on the following cases about  $\uparrow_{k+1} P(\mathbf{x})$ , such that if  $\uparrow_{k+1} P(\mathbf{x})$  is:

- $\exists x\psi$ , then for a valid justification of  $\phi$ , there must be some  $a$  in the domain of the underlying structure to establish  $\phi[x/a]$ . Hence, we must have that  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}] = \exists x\phi$ ;
- $\forall x\psi$ , then for a valid justification of  $\phi$ , in addition to the requirement that  $\exists x\phi$  must hold, we must also have that  $\forall x\psi$  holds as well, i.e.,  $\forall x\psi \wedge \exists x\phi$ ;
- $(\psi \vee \xi)$  for some  $\xi$ , then it must be that  $\phi$  is at least true. Hence, we just retain  $\phi$  so that  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}] = \phi$ ;
- $(\psi \wedge \xi)$  for some  $\xi$ , then in addition to the requirement that  $\phi$  itself must hold,  $\xi$  must also hold as well, i.e.,  $\xi \wedge \phi$  (similarly to the case of the universal quantifier “ $\forall x\psi$ ”);
- $(\psi \rightarrow \xi)$  (i.e.,  $\psi$  is in the *antecedent*), then there is no constructive reason to assume the existence of  $\psi$ , therefore, we must have that  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}] = \perp$ ;
- $(\xi \rightarrow \psi)$  (i.e.,  $\psi$  is in the *consequent*), then in the spirit of constructivism, it must be that  $\psi$  exists (or constructed) due to  $\xi$  and such that  $P(\mathbf{y})$  is not used in the establishment  $\xi$ , therefore we must have that  $\phi \wedge \xi^{<P(\mathbf{y})}$ .

### Example 2

Assume  $\varphi$  to be the formula as follows:

$$\varphi = \forall x(P(x) \rightarrow \exists y[P(x) \vee \forall z(Q(x, z) \wedge R(x, y))]).$$

Then for the atom  $R(x, y) \in \text{SUBF}[\varphi]$  and tuple  $y_1 y_2$  of fresh distinct variables, we have:

$$\begin{aligned}
 \uparrow_0 R(x, y)[y_1, y_2] &= y_1 = x \wedge y_2 = y; \\
 \uparrow_1 R(x, y)[y_1, y_2] &= Q(x, z) \wedge \uparrow_0 R(x, y)[y_1, y_2]; \\
 \uparrow_2 R(x, y)[y_1, y_2] &= \forall z(Q(x, z) \wedge R(x, y)) \wedge \exists z(\uparrow_1 R(x, y)[y_1, y_2]); \\
 \uparrow_3 R(x, y)[y_1, y_2] &= \uparrow_2 R(x, y)[y_1, y_2] \text{ (i.e., since we encountered an “} \vee \text{”)}; \\
 \uparrow_4 R(x, y)[y_1, y_2] &= \exists y(\uparrow_3 R(x, y)[y_1, y_2]); \\
 \uparrow_5 R(x, y)[y_1, y_2] &= P(x) \wedge P(x) < \widehat{R}(y_1, y_2) \wedge \uparrow_4 R(x, y)[y_1, y_2] \\
 \uparrow_6 R(x, y)[y_1, y_2] &= \varphi \wedge \exists x(\uparrow_5 R(x, y)[y_1, y_2]) \\
 &= \varphi \wedge \exists x[P(x) \wedge P(x) < \widehat{R}(y_1, y_2) \\
 &\quad \wedge \exists y(\forall z(Q(x, z) \wedge R(x, y)) \wedge \exists z(Q(x, z) \wedge y_1 = x \wedge y_2 = y))],
 \end{aligned}$$

where  $P(x) < \widehat{R}(y_1, y_2)$  denotes the formula  $\leq_{PR}(x, y_1, y_2) \wedge \neg \leq_{RP}(y_1, y_2, x)$  and  $\uparrow_\infty R(x, y)[y_1, y_2] = \uparrow_6 R(x, y)[y_1, y_2]$  is the fixpoint. Then  $\forall y_1 y_2 (R(y_1, y_2) \rightarrow \uparrow_\infty R(x, y)[y_1, y_2])$  now expresses the explicit constructor for all extents of  $R$  under  $\varphi$ .  $\square$

### 3.2 Constructive Models

We now show how the notion of explicit construction is used in constructive circumscription and where the main idea follows in the lines of the ordered completion (Asuncion et al. 2012).

*Definition 4 (Explicit construction formula)*

Let  $\varphi$  be an FO sentence and  $\mathbf{P} = P_1 \dots P_n$  its tuple of distinctive intensional predicates from the set  $\mathcal{P}_{int}$ . In addition, let  $\mathbf{T}$  be the tuple of distinctive comparison predicates from the set  $\{\leq_{PQ} \mid P, Q \in \mathcal{P}_{int}\}$ . Then by  $\varphi_{\mathbf{P}}^{\text{EC}}$  (“EC” for *explicit construction*), we denote the following FO sentence:

$$\bigwedge_{1 \leq i \leq n} \forall \mathbf{y} (P_i(\mathbf{y}) \rightarrow \bigvee_{P_i(\mathbf{x}) \in \text{SUBF}[\varphi]} \uparrow_\infty P_i(\mathbf{x})[\mathbf{y}]) \wedge \text{TRANS}(\mathbf{T}) \quad (7)$$

where  $\text{TRANS}(\mathbf{T})$  denotes the sentence

$$\bigwedge_{1 \leq i, j, k \leq n} \forall \mathbf{x} \mathbf{y} \mathbf{z} (\leq_{P_i P_j}(\mathbf{x}, \mathbf{y}) \wedge \leq_{P_j P_k}(\mathbf{y}, \mathbf{z}) \rightarrow \leq_{P_i P_k}(\mathbf{x}, \mathbf{z})), \quad (8)$$

which encodes that the comparison atoms satisfy a notion of transitivity.  $\square$

*Definition 5 (Constructive circumscription formula)*

Let  $\varphi$  be an FO sentence,  $\mathbf{P}$  be the tuple of its distinctive intensional predicates, and  $\mathbf{U}$  and  $\mathbf{T}$  be two disjoint tuples of predicate variables of the same length as  $\mathbf{P}$  and comparison predicates respectively. Then by  $\text{C-CIRC}_{\mathbf{P}}[\varphi]$  (“C-CIRC” for *constructive circumscription*), we denote the following SO sentence:

$$\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}} \wedge \neg \exists \mathbf{U} \mathbf{T} [\mathbf{U} < \mathbf{P} \wedge (\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}})(\mathbf{U} \mathbf{T})], \quad (9)$$

where  $(\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}})(\mathbf{U} \mathbf{T})$  denotes the formula obtained from  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}}$  by replacing all predicates from  $\mathbf{P}$  and all comparison predicates by those corresponding ones from  $\mathbf{U}$  and  $\mathbf{T}$  respectively.  $\square$

*Definition 6 (Constructive models)*

For an FO formula  $\varphi$  with tuple  $\mathbf{P}$  of intensional predicates  $\mathcal{P}_{int}$ , a  $\tau(\varphi)$ -structure  $\mathcal{M}$  is a *constructive model* of  $\varphi$  iff  $\mathcal{M}$  can be expanded to a model of  $\mathbf{C}\text{-CIRC}_{\mathbf{P}}[\varphi]$ .  $\square$

Intuitively, in Definition 5, formula  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}}$  captures the explicit derivation order for the underlying reasoning in  $\varphi$  while formula  $\neg\exists\mathbf{UT}[\mathbf{U} < \mathbf{P} \wedge (\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}})(\mathbf{UT})]$  imposes the minimality requirement on all intensional predicates  $\mathbf{P}$ . Therefore, constructive models in Definition 6, i.e., models of  $\mathbf{C}\text{-CIRC}_{\mathbf{P}}[\varphi]$ , are the  $\mathbf{P}$ -minimal models of  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}}$  with the comparison predicates allowed to vary.

*Example 3*

Let  $\varphi$  be the following propositional formula:  $[(p \rightarrow q) \rightarrow r] \rightarrow p \wedge (q \rightarrow r) \wedge (r \rightarrow q)$ . Then it can be shown that  $\varphi$  has the two stable models  $\{\}$  and  $\{p\}$ , while  $\{\}$  is its only constructive model. Indeed, since  $\varphi_{pqr}^{\text{EC}}$  from Definition 4 is <sup>4</sup>:

$$[p \rightarrow ((p \wedge \widehat{p} < \widehat{p} \rightarrow q) \rightarrow (r \wedge \widehat{r} < \widehat{p}))] \wedge [q \rightarrow r \wedge \widehat{r} < \widehat{q}] \wedge [r \rightarrow q \wedge \widehat{q} < \widehat{r}] \quad (10)$$

$$\wedge \bigwedge_{t_1, t_2, t_3 \in \{p, q, r\}} (\leq_{t_1 t_2} \wedge \leq_{t_2 t_3} \rightarrow \leq_{t_1 t_3}), \quad (11)$$

then through the transitivity axioms (11),  $\varphi_{pqr}^{\text{EC}}$  would imply that  $p$ ,  $q$  and  $r$  are all circularly justified. Hence, none of them can be in any model of  $\varphi_{pqr}^{\text{EC}}$ . Incidentally, if we conjunct  $\varphi$  with  $\neg p$ , then  $\varphi \wedge \neg p$  will have  $\{p\}$  as both its only minimal and stable models, which still happens to be circularly justified.  $\square$

**4 Properties of Constructive Models**

Now we show that constructive models satisfies the following three desirable properties: *anti-chain*, *supportedness*, and *non-circular derivation*. Given a structure  $\mathcal{M}$  and a tuple of predicate symbols  $\mathbf{P}$ ,  $\mathcal{M}|_{\mathbf{P}}$  denotes the *restriction* of  $\mathcal{M}$  to those predicates whose symbols are mentioned in  $\mathbf{P}$ . We also denote  $\mathcal{M}'|_{\mathbf{P}} \subset \mathcal{M}|_{\mathbf{P}}$  if  $P^{\mathcal{M}'} \subset P^{\mathcal{M}}$  for each predicate symbol  $P$  of  $\mathbf{P}$ .

*Theorem 1*

The constructive models have the anti-chain property with respect to  $\mathbf{P}$ . That is, if  $\mathcal{M}$  is a constructive model of  $\varphi$ , then there does not exist an  $\mathcal{M}'$  where  $\mathcal{M}'|_{\mathbf{P}} \subset \mathcal{M}|_{\mathbf{P}}$  such that  $\mathcal{M}'$  is also a constructive model of  $\varphi$ .  $\square$

Now we try to provide a logical characterization on the anti-chain property. Given an SO formula  $\Phi$  with a tuple of intensional predicates  $\mathbf{P}$ , we define  $\Phi_{\mathbf{P}}^{\text{AC}}$  (AC for anti-chain) as follows:  $\forall\mathbf{UV}(\mathbf{U} < \mathbf{P} \rightarrow \neg\Phi(\mathbf{UV}))$ , where  $\mathbf{U}$  is a tuple of fresh predicate variables of the same length as  $\mathbf{P}$ , and  $\mathbf{V}$  is the tuple of predicate variables corresponding to the tuple of predicates in  $\Phi$  that are allowed to vary <sup>5</sup>. Then  $\Phi_{\mathbf{P}}^{\text{AC}}$  captures those models of  $\Phi$  that are  $\mathbf{P}$ -minimal with respect to the varying and the fixed extensional predicates.

*Proposition 1*

$\mathbf{C}\text{-CIRC}_{\mathbf{P}}[\varphi] \models [\mathbf{C}\text{-CIRC}_{\mathbf{P}}[\varphi]]_{\mathbf{P}}^{\text{AC}}$ .  $\square$

<sup>4</sup> We simplified for the fact that  $\varphi_{pqr}^{\text{EC}}$  will be conjuncted with  $\varphi$ .

<sup>5</sup> In the case of constructive circumscription,  $\mathbf{V}$  will correspond to those newly introduced auxiliary comparison predicates, while for circumscription, stable models and FLP,  $\mathbf{V}$  is empty, as no new predicate was introduced in their semantics definitions.



*Definition 7 (Supportedness formula)*

For an FO formula  $\varphi$ , we define a formula, denoted by  $\varphi_{\mathbf{P}}^{\text{SUP}}$  (“SUP” for “support”), as follows<sup>6</sup>:

$$\bigwedge_{1 \leq i \leq n} \forall \mathbf{y} (P_i(\mathbf{y}) \rightarrow \bigvee_{P_i(\mathbf{x}) \in \text{SUBF}[\varphi]} \uparrow_{\infty}^{\text{SUP}} P_i(\mathbf{x})[\mathbf{y}]), \quad (12)$$

where  $\uparrow_{\infty}^{\text{SUP}} P_i(\mathbf{x})[\mathbf{y}]$  is defined as in  $\uparrow_{\infty} P_i(\mathbf{x})[\mathbf{y}]$  via Definition 3 except that, we do not enforce the comparison atoms in the case of the implication ( $\xi \rightarrow \psi$ ) into  $\xi$  (hence, no need for the transitivity axioms), i.e., while we have  $\uparrow_{k+1} P(\mathbf{x})[\mathbf{y}] = \phi \wedge \xi^{<P(\mathbf{y})}$  in Definition 3, we simply have  $\uparrow_{k+1}^{\text{SUP}} P(\mathbf{x})[\mathbf{y}] = \phi \wedge \xi$  for  $\uparrow_{k+1}^{\text{SUP}} P(\mathbf{x})[\mathbf{y}]$ .  $\square$

Generally speaking, we can view  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{SUP}}$  as a generalization of the notion of Clark’s completion to arbitrary FO formulas. In fact,  $\varphi_{\mathbf{P}}^{\text{SUP}}$  captures the notion of a “supported model,” i.e., there is an explanation for an extent of a predicate in a model of  $\varphi$ .

*Proposition 2*

$$\text{C-CIRC}_{\mathbf{P}}[\varphi] \models \varphi \wedge \varphi_{\mathbf{P}}^{\text{SUP}}. \quad \square$$

As we mentioned in the previous section, formula  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}}$  captures the explicit derivation order for the reasoning in  $\varphi$  given  $\mathbf{P}$  as the tuple of intensional predicates. Furthermore, from the construction of  $\varphi_{\mathbf{P}}^{\text{EC}}$  (i.e., Definitions 3 and 4), it is observed that  $\varphi \wedge \varphi_{\mathbf{P}}^{\text{EC}}$  precisely captures non-circular derivation order in the sense that an extent  $P(\mathbf{a})$  of  $P$  will never be used to establish the truth of itself.

Now let  $\exists \mathbf{T} \varphi_{\mathbf{P}}^{\text{EC}}$  be the formula obtained from  $\varphi_{\mathbf{P}}^{\text{EC}}$  by replacing all comparison predicates with the tuple  $\mathbf{T}$  of comparison predicate variables accordingly. Then  $\varphi \wedge \exists \mathbf{T} \varphi_{\mathbf{P}}^{\text{EC}}$  encodes the non-circular derivation order under  $\tau(\varphi)$ .

*Proposition 3*

$$\text{C-CIRC}_{\mathbf{P}}[\varphi] \models \varphi \wedge \exists \mathbf{T} \varphi_{\mathbf{P}}^{\text{EC}}. \quad \square$$

## 5 Comparisons with Circumscription, Stable Models and FLP

In this section, we provide some detailed comparisons of C-CIRC with CIRC, SM, and FLP. Firstly, we have the following result showing that C-CIRC, CIRC, and SM coincide on implication free formulas.

*Proposition 4*

Let  $\varphi$  be an implication free FO sentence and  $\mathbf{P}$  a tuple of intensional predicates of  $\varphi$ , we have  $\text{Mod}(\text{C-CIRC}_{\mathbf{P}}[\varphi]) \upharpoonright_{\tau(\varphi)} = \text{Mod}(\text{CIRC}_{\mathbf{P}}[\varphi]) = \text{Mod}(\text{SM}_{\mathbf{P}}[\varphi])$ <sup>7</sup>.  $\square$

In the following, we will assume that  $\varphi$  is an arbitrary FO sentence, and  $\mathbf{P}$  is the tuple of intensional predicates of  $\varphi$ .

*Theorem 2*

$$\text{CIRC}_{\mathbf{P}}[\varphi] \models [\text{CIRC}_{\mathbf{P}}[\varphi]]_{\mathbf{P}}^{\text{AC}}, \text{CIRC}_{\mathbf{P}}[\varphi] \not\models \varphi \wedge \varphi_{\mathbf{P}}^{\text{SUP}}, \text{ and } \text{CIRC}_{\mathbf{P}}[\varphi] \not\models \varphi \wedge \exists \mathbf{T} \varphi_{\mathbf{P}}^{\text{EC}}. \quad \square$$

<sup>6</sup> Note that the following formula is similar to  $\varphi_{\mathbf{P}}^{\text{EC}}$  apart from where we have  $\uparrow_{\infty}^* P_i(\mathbf{x})[\mathbf{y}]$  instead of  $\uparrow_{\infty} P_i(\mathbf{x})[\mathbf{y}]$ , and where we do not conjunct it with  $\text{TRANS}(\mathbf{T})$ .

<sup>7</sup> Here  $\text{Mod}(\text{C-CIRC}_{\mathbf{P}}[\varphi]) \upharpoonright_{\tau(\varphi)}$  denotes the set of all models of  $\text{C-CIRC}_{\mathbf{P}}[\varphi]$  restricted on  $\tau(\varphi)$ .

*Theorem 3*

$\text{SM}_{\mathbf{P}}[\varphi] \not\models [\text{SM}_{\mathbf{P}}[\varphi]]_{\mathbf{P}}^{\text{AC}}$ ,  $\text{SM}_{\mathbf{P}}[\varphi] \models \varphi \wedge \varphi_{\mathbf{P}}^{\text{SUP}}$ , and  $\text{SM}_{\mathbf{P}}[\varphi] \not\models \varphi \wedge \exists \mathbf{T} \varphi_{\mathbf{P}}^{\text{EC}}$ .  $\square$

Theorem 3 indicates that in general, the stable model semantics does not enforce non-circular derivation order as compared to constructive models, although it does embed “supportedness”. This is also evident from Example 3 where  $\varphi$  has the two stable models  $\{\}$  and  $\{p\}$ , and where the stable model  $\{p\}$  complies with the circular derivation through  $[(p \rightarrow q) \rightarrow r] \rightarrow p$ .

*Theorem 4*

Let  $\Pi$  be a general logic program and assume for all rules  $B \rightarrow H \in \Pi$  that  $H$  contains no implications<sup>8</sup>. Then we have that  $\text{FLP}_{\mathbf{P}}[\widehat{\Pi}] \models [\text{FLP}_{\mathbf{P}}[\widehat{\Pi}]]_{\mathbf{P}}^{\text{AC}}$ ,  $\text{FLP}_{\mathbf{P}}[\widehat{\Pi}] \models \widehat{\Pi} \wedge \widehat{\Pi}_{\mathbf{P}}^{\text{SUP}}$ , and  $\text{FLP}_{\mathbf{P}}[\widehat{\Pi}] \not\models \widehat{\Pi} \wedge \exists \mathbf{T} \widehat{\Pi}_{\mathbf{P}}^{\text{EC}}$ .  $\square$

## 6 Concluding Remarks

In this paper, with the anti-chain, supportedness, and non-circular derivation being three important and desirable properties for most nonmonotonic reasoning tasks, we showed that among circumscription, the general theory of stable models, and the FLP semantics, our constructive circumscription is the only one that satisfies all these rational properties. In our full paper, we further show that our constructive circumscription may be extended to aggregates and choice rules<sup>9</sup> without compromising its reasoning features.

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<sup>8</sup> Without loss of generality, since the implications in  $H$  are treated classically, one can transform  $H$  into a classically equivalent formula using only connectives  $\vee$ ,  $\wedge$  and  $\neg$ .

<sup>9</sup> We show that even in the presence of choice rules, the anti-chain property is still an important rational property when considering arbitrary FO formulas because, since choice rules are usually associated with choice predicates, it can be shown that the anti-chain property still do not hold for the remaining non-choice predicates.