Dynamic Auction for Efficient Competitive Equilibrium under Price Rigidities*

Junwu Zhu[1,2] and Dongmo Zhang[1]

[1] Intelligent Systems Laboratory, School of Computing and Mathematics, University of Western Sydney, Australia
[2] School of Information Engineering, Yangzhou University, China

Abstract. In an auction market where the price of each selling item is restricted to an admissible interval (price rigidities), a Walrasian equilibrium usually fails to exist. Dreze (1975) introduced a variant concept of Walrasian equilibrium based on rationing systems, named constrained Walrasian equilibrium, for modelling an economy with price rigidities. Talman and Yang (2008) further refined the concept and proposed a dynamic auction procedure that converges to a constrained Walrasian equilibrium. However, a constrained Walrasian equilibrium does not guarantee market efficiency. In other words, a constrained Walrasian equilibrium allocation does not necessarily lead to the best market value. In this paper, we introduce a concept of competitive equilibrium by weakening the concept of constrained Walrasian equilibrium and devise an dynamic auction procedure that generates an efficient competitive equilibrium.

1 Introduction

Auctions have been widely used for discovering market-clearing prices and efficient allocations [1]. However, in many market situations, the price of an item cannot be fully determined by its market. There are certain exogenous reasons that could cause the price of a selling item not completely flexible. For instance, price ceilings and floors in stock markets to prevent breakdown; price controls to reduce inflation or deflation; and imposing upper prices to protect low-income buyers [2,3,4]. Such a phenomenon is normally referred to as *price rigidities* in economics.

In a market with price rigidities, certain rationing mechanism is normally needed to facilitate the distribution of commodities among agents in additional to the price leverage. Dreze (1975) introduced a variant concept of Walrasian equilibrium based on rationing, named constrained Walrasian equilibrium, for economies with price rigidities [2]. Talman and Yang (2008) further refined the concept and proposed a dynamic auction procedure that produces a constrained Walrasian equilibrium outcome in a finite number of steps [4]. However, as we will show in this paper, a constrained equilibrium under Talman and Yang's definition does not guarantee market efficiency.

At first glance, it seems impossible for a dynamic auction procedure to achieve market efficiency in a market with price rigidities because the market value of an item over

* This work was partly supported by the Australian Research Council through project LP0883646.

D. Wang and M. Reynolds (Eds.): AI 2011, LNAI 7106, pp. 809–818, 2011.
© Springer-Verlag Berlin Heidelberg 2011

its price upper limit can never be discovered by any auction procedure. However, if we count bidders' contributions to the market value within the constraint price intervals, market efficiency will be achievable.

Similar to Talman and Yang (2008), we study the market situations where the following conditions hold:

- The commodities to be sold are heterogeneous and indivisible, such as cars and houses;
- Each buyer can buy at most one item at each auction;
- The price restriction on each item is represented by an interval, the lower bound and the upper bound, which is given to the auctioneer as a reservation at the beginning of an auction procedure.

The rest of the paper is organised in the following. Section 2 sets up the model of the underlying markets. Section 3 presents our dynamic auction procedure and prove that the procedure can find an efficient competitive equilibrium in a finite number of steps. Section 4 gives an example of how the procedure runs. Finally we conclude the work with brief remarks on the related work.

2 The Market Model

Consider a market situation where a seller wishes to sell a finite set of items to a finite number of buyers. Each item is indivisible and the items are heterogeneous. Each buyer has a private value over each item. Formally, let X be the set of items on offer, N the set of buyers, and v^i the value function of buyer i $(i \in N)$. We assume that the seller values each item in X at zero. We also assume that among the items in X, there is a specific item, called the dummy item, which value is zero to each buyer and the seller. For sake of simplicity, we let $N = \{1, 2, \cdots, n\}$ and $X = \{0, 1, \cdots, m\}$, where item 0 represents the dummy item.

We assume that each buyer i has an integer value function, i.e., $v^i : X \to \mathbb{Z}^+$, which assigns each item $j \in X$ an integer $v^i(j)$ (in the unit of money) with $v^i(0) = 0$.

A *price vector* $\mathbf{p}$ is a function $\mathbf{p} : X \to \mathbb{Z}^+$ that assigns a non-negative integer to each item in X. For each $j \in X$, we write p_j, instead of $\mathbf{p}(j)$, to indicate the price of item j under the price vector $\mathbf{p}$.

As we have mentioned in the previous section, we will consider in this paper the problem of price discovery under price rigidities. We assume that the price of each item $j \in X$ is restricted to an interval $[\underline{p}_j, \overline{p}_j]$, where $\underline{p}_j$ and $\overline{p}_j$ are integers and $0 \le \underline{p}_j \le \overline{p}_j < +\infty$. Specifically, we assume that $\underline{p}_0 = \overline{p}_0 = 0$, which means that the price of the dummy item can only be zero. $\overline{p}_j = +\infty$ means that there is no upper bound limit of price to item j. We say that a price vector $\mathbf{p}$ is *admissible* if $\underline{p}_j \le p_j \le \overline{p}_j$ for all $j \in X$.

Traditionally, the following defines the *demand correspondence* of bidder i at price vector $\mathbf{p}$:

$$D^i(\mathbf{p}) = \{j \in X \mid v^i(j) - p_j \ge v^i(k) - p_k, \forall k \in X\}. \tag{1}$$

The following defines all the items that the bidder i would demand thus we call it the *demand set* of bidder i at $\mathbf{p}$:

$$M^i(\mathbf{p}) = \{j \in X \setminus \{0\} \mid v^i(j) \ge p_j\} \tag{2}$$

In a competitive market, it is possible that one item is demanded by more than one bidders. The following notation represents all the items that are demanded by more than one buyers at price vector $\mathbf{p}$, called *over-demanded set*:

$$O(\mathbf{p}) = \{j \in X : \exists i, i' \in N(i \neq i' \ \&\ j \in M^i(\mathbf{p}) \cap M^{i'}(\mathbf{p}))\} \tag{3}$$

Following the assumption of Talman and Yang [4], we assume in this paper that each buyer can only receive one item and each item, except the dummy item, can only be allocated to one buyer. Based on the assumption, an *allocation* of X can be represented as a function $\pi : N \to X$ that satisfies the following condition:

– If $\pi(i) = \pi(i')$ and $i \neq i'$, then $\pi(i) = \pi(i') = 0$.

Traditionally, an allocation π^* being efficient means that it gives the best market value, i.e., for any allocation π of X in N,

$$\sum_{i \in N} v^i(\pi^*(i)) \geq \sum_{i \in N} v^i(\pi(i))$$

However, such a traditional definition of efficiency is not applicable to the markets with price rigidities because the market value of an item over its price upper limit can never be discovered by any auction procedure. For this reason, we redefine the concept of market efficiency as follows.

Definition 1. *Let π be an allocation π of X, the market value of π at price vector $\mathbf{p}$ under price rigidities is $\sum_{i \in N}(min(v^i(\pi(i)), \bar{p}_{\pi(i)}) - \underline{p}_{\pi(i)})$, where v^i is the value function of bidder i.*

Note that the market rule means the totally value the market generates. No value can be generated under the lower bound.

Definition 2. *An allocation π^* of X is* efficient *if, for any allocation π of X,*

$$\sum_{i \in N}(\min\{v^i(\pi^*(i)), \bar{p}_{\pi^*(i)}\} - \underline{p}_{\pi^*(i)}) \geq \sum_{i \in N}(\min\{v^i(\pi(i)), \bar{p}_{\pi(i)}\} - \underline{p}_{\pi(i)}) \tag{4}$$

In economics, a rationing system describes a set of market rules. Formally, a *rationing system* $R = (R^i_j)_{i \in N, j \in X}$ is a $|N| \times |X|$ matrix, which element has a value either 1 or 0. For each $i \in N$ and $j \in X$, $R^i_j = 1$ means that buyer i has right to buy item j while $R^i_j = 0$ indicates that buyer i is prohibited to buy item j. With a rationing system R, the demand correspondence can be re-defined as follows:

$$D^i(\mathbf{p}, R) = \{j \in X \mid R^i_j = 1 \text{ and } \min\{v^i(j), \bar{p}_j\} - p_j \geq \max\{\min\{v^i(h), \bar{p}_h\} - p_h \mid R^i_h = 1\}\} \tag{5}$$

Based on a rationing system, Talman and Yang [4] gave the following variation of Walrasian equilibrium:

Definition 3. [4] *A tuple $(\mathbf{p}^*, \pi^*, R^*)$ is a constrained Walrasian equilibrium if*

1. *π^* is an allocation, $\mathbf{p}^*$ is an admissible price vector, and R^* is a rationing system;*
2. *$\pi^*(i) \in D_i(\mathbf{p}^*, R^*)$ for all $i \in N$;*

3. $p_j^* = \underline{p}_j$, *if* $\pi^*(i) \neq j$ *for all* $i \in N$;
4. $p_j^* = \bar{p}_j$ *and* $\pi(h) = j$ *for some* $h \in N$ *if* $R_j^{*i} = 0$ *for some* $i \in N$;
5. $j \in D^i(\mathbf{p}^*, R_{-j}^{*i})$ *if* $R_j^{*i} = 0$.

where R_{-j}^{*i} *denote that* R_j^i *is being ignored and bidder* i *is allowed to demand item* j *(see [4]).*

Talman and Yang devised a dynamic auction procedure that can generate a constrained Walrasian equilibrium [4].

Theorem 1. [4] *There exists at least one constrained Walrasian equilibrium in the model under price rigidities.*

Unfortunately, the allocation of a constrained Walrasian equilibrium is not necessarily efficient.

Example 1. Suppose that $N = \{1, 2, 3\}$ and $X = \{0, 1, 2, 3\}$. The lower and upper bound of prices are $\underline{\mathbf{p}} = \{0, 0, 0, 0, \}$ and $\overline{\mathbf{p}} = \{0, 10, 10, 30\}$. Bidders' values are given as follows:

	item 0	item 1	item 2	item 3
Bider 1	0	6	7	38
Bider 2	0	8	6	40
Bider 3	0	0	0	28

There are two constrained Walrasian equilibria. The price vector of both equilibria is $\mathbf{p}^* = (0, 0, 30)$. The allocation and rationing system of the first equilibrium is $\pi^* = (2, 3, 1)$ and $R^* = ((1, 1, 0), (1, 1, 1), (1, 1, 0))$. The equilibrium gives a market value $7 + 30 + 0 = 37$. The other equilibrium is $\mathbf{p}'^* = (0, 0, 30)$, $\pi'^* = (3, 1, 2)$ and $R'^* = ((1, 1, 1), (1, 1, 0), (1, 1, 0))$. The market value of this equilibrium is $30 + 8 + 0 = 38$. However, if the allocation is $(2, 1, 3)$, which is not a constrained Walrasian equilibrium allocation, the market value can be $7 + 8 + 28 = 44$.

The above example shows that an efficient allocation may not be a constrained Walrasian equilibrium allocation. Therefore, if our target is to get an efficient allocation, we have to weaken the concept of constrained Walrasian equilibrium. The following definition of competitive equilibrium is actual a weak version Talman and Yang's concept of constrained Walrasian equilibrium:

Definition 4. *A* competitive equilibrium with rationing *is a triple* $(\mathbf{p}^*, \pi^*, R^*)$*, where* $\mathbf{p}^*$ *is an admissible price vector,* π^* *is an allocation and* R^* *is a rationing scheme at* $\mathbf{p}^*$ *such that*

1. $\pi^*(i) \in D^i(\mathbf{p}^*, R^*)$ *for all* $i \in N$.
2. $p_j^* = \underline{p}_j$, *if* $\pi^*(i) \neq j$ *for all* $i \in N$;
3. $\min(v^i(j), \bar{p}_j) \geq \min(v^{i'}(j), \bar{p}_j)$ *and* $\pi^*(i') = j$ *if* $R_j^{*i} = 0$ *for some* $i \in N$.

A competitive equilibrium with rationing $(\mathbf{p}^*, R^*, \pi^*)$ *is efficient if the allocation* π^* *is efficient.*

The first two conditions are exactly the same as Talman and Yang's definition. The third reads a bid strange: only the bidders who has the highest value on the item get rationed. In fact, rationing is a privilege of the auctioneer to govern a market and only those bidders who have high valuation on an item need to be rationed (because they are more likely to get the goods). For instance, a government could ban high-income people applying for public housing.

Lemma 1. *Each constrained Walrasian equilibrium is a competitive equilibrium with rationing.*

Proof. Assume that $(\mathbf{p}^*, R^*, \pi^*)$ is a constrained Walrasian equilibrium. Then the first two conditions for a competitive equilibrium with a rationing system is satisfied. Now we prove that the third condition holds. Let $R_j^{*i} = 0$ for some $i \in N$. According to Condition 4 for a constrained Walrasian equilibrium, we have $p_j^* = \bar{p}_j$ and $\pi(h) = j$ for some $h \neq i$. It turns out that $min(v^i(j), \bar{p}_j) = \bar{p}_j$ and $min(v^h(j), \bar{p}_j) = \bar{p}_j$. Consequently, $min(v^i(j), \bar{p}_j) = min(v^h(j), \bar{p}_j)$, as desired. □

As we have shown in above example, an efficient competitive equilibrium with rationing is not necessarily a constrained Walrasian equilibrium.

3 Dynamic Auction Procedure under Price Rigidities

In this section, we will introduce a dynamic auction procedure that can generate an efficient competitive equilibrium with rationing.

Given a set $N = \{1, 2, \cdots, n\}$ of bidders and a set $X = \{0, 1, 2, \cdots, m\}$ of items on offer, where 0 is a dummy item which can be allocated to more than one bidders. $\bar{p}$ and $\underline{p}$ are the upper price bound and the lower price bound respectively. The dynamic auction procedure consists of the following steps:

Step 1. Set the initial price vector $\mathbf{p} := \underline{\mathbf{p}}$ and the initial rationing scheme $R_j^i = 1$ for all $i \in N, j \in X$. Let $S = (S_{i,j})_{i \in N, j \in X}$ be a $n \times m$ matrix initiated as follows:

$$S = \begin{pmatrix} 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \end{pmatrix} \tag{6}$$

In this matrix, the rows represent the bidders and the column for items. The elements are initiated by $S_{i,j} := -\infty$ for all $i \in N, j \in X$ except zero for the dummy item.

Step 2. Auctioneer announces the price vector $\mathbf{p}$ and invites all the buyers to submit their demand set $M^i(\mathbf{p})$. For all $j \in M^i(\mathbf{p})$, $S_{i,j} := p_j$.

Step 3. Calculate over-demanded set $O(\mathbf{p})$. If $O(\mathbf{p}) \neq \emptyset$ and $p_j < \bar{p}_j$ for all $j \in O(\mathbf{p})$, then go to Step 4. Otherwise go to Step 5.

Step 4. For all $j \in O(\mathbf{p})$ such that $p_j < \bar{p}_j$, let $p_j := p_j + 1$. Go back to Step 2.

Step 5. Construct a weighted bipartite graph $G = (N \cup X, E, W)$, where
- $E \subseteq N \times X$ such that $e_{i,j} \in E$ iff $S_{i,j} \neq -\infty$ for all $i \in N, j \in X$
- $W : E \to \mathbb{Z}$ such that $W(e_{i,j}) = S_{i,j} - \underline{p}_j$ for each $e_{i,j} \in E$.

Step 6. Let $\Omega \subseteq E$ be a maximum weighted bipartite matching in G[1].

Step 7. For each $i \in N$, if $e_{i,j} \in \Omega$, let $\pi^*(i) = j$ and $p^*_j = S_{i,j}$; Meanwhile, for each $k \in N$ such that $k \neq i$ and $S_{k,\pi^*(i)} \geq S_{i,\pi^*(i)}$, let $R^{*k}_{\pi^*(i)} := 0$.

Since each item has a finite price upper bound, the above dynamic auction procedure terminates in finite number of steps. Let $(\mathbf{p}^*, R^*, \pi^*)$ be the outcome of the procedure when it terminates.

Lemma 2. $\pi^*(i) \in D^i(\mathbf{p}^*, R^*)$.

Proof. Let $\pi^*(i) = j$. Firstly, $S_{i,j} \neq -\infty$ because no edge links between i and j in the associated bipartite graph. For all $k \in N$, if $k \neq i$ and $S_{k,j} \geq S_{i,j}$, then $R^{*k}_j = 0$. Assume that there is a $h \in X$ such that $\min\{v^i(h), \bar{p}_h\} - p^*_h > \min\{v^i(j), \bar{p}_j\} - p^*_j$ and $R^i_h = 1$. If $p^*_h < \bar{p}_h$, it means that only i bids for h at price p_h. We change the matching from $\pi^*(i) = j$ to $\pi^*(i) = h$ and keep the other allocation unchanged. We can then increase the weight of the matching, which contradicts the fact that π^* is a maximum weighted matching. If $p^*_h = \bar{p}_h$, then we have $v^i(j) < p^*_j$. It implies that $S_{i,j} < p^*_j$. By the construction of the rationing system, $R^{*k}_j = 0$ for all $k \neq i$ and $S_{k,j} >= S_{i,j}$. In other words, $v^i(j) - p^*_j \geq \max\{v^i(h) - p^*_h \mid R^i_h = 1\}$. Therefore we have $\pi^*(i) \in D^i(p^*, R^*)$. □

Theorem 2. (p^*, R^*, π^*) *is an efficient competitive equilibrium.*

Proof. Lemma 2 has shown that the dynamic auction mentioned above can yield a competitive equilibrium (p^*, R^*, π^*). We now prove that π^* is an efficient allocation.

π^* is the maximum weighted matching of the weighted graph $G = (N \cup X, E, W)$ as defined in the auction procedure. Assume that there is $\pi^{'}$ is efficient allocation, which obviously satisfies the following inequality:

$$\sum_{i\in N}(min\{v^i(\pi^{'}(i)), \bar{p}_{\pi^{'}(i)}\} - \underline{p}_{\pi^{'}(i)}) > \sum_{i\in N}(min\{v^i(\pi^*(i)), \bar{p}_{\pi^*(i)}\} - \underline{p}_{\pi(i)}) \quad (7)$$

Note that the allocation π' also determines a matching in the weighted bipartite graph unless there is an i such that $S_{i,\pi'(i)} = -\infty$. In this case, $v_i(\pi'(i)) < \underline{p}_{\pi(i)}$. Now we define a new allocation π'' such that $\pi''(i) = 0$ and $\pi''(j) = \pi'(j)$ for all $j \neq i$. It turns out that π'' can implement more market value than π', which contradicts to the assumption. On the other hand, π' cannot be a maximum weighted matching of G because otherwise π will not be a maximum weighted matching of G. Therefore $(\mathbf{p}^*, R^*, \pi^*)$ is an efficient competitive equilibrium. □

[1] We omit the algorithm for finding a maximum weighted matching in a bipartite graph. In fact, any maximum weighted bipartite matching algorithm is applicable. The reader is referred to the algorithm in [5].

4 Calculation and Comparison

To compare our auction procedure with Talman and Yang's, we use the same example that has been used in [4] to demonstrate how to calculate an efficient competitive equilibrium with rationing by using the dynamic auction procedure introduced in the previous section.

Example 2. Suppose that there are five bidders $N = \{a, b, c, d, e\}$ and five items $X = \{0, 1, 2, 3, 4\}$ in a market, where 0 is a dummy item and the others are real items. The lower and upper price vectors are $\underline{\mathbf{p}} = (0, 5, 4, 1, 5)$ and $\bar{\mathbf{p}} = (0, 6, 6, 4, 7)$, respectively. Bidders' values are given by the following table.

Item	dummy	1	2	3	4
Bidder a	0	4	3	5	7
Bidder b	0	7	6	8	3
Bidder c	0	5	5	7	7
Bidder d	0	9	4	3	2
Bidder e	0	6	2	4	10

Initially, we set $\mathbf{p} = \underline{\mathbf{p}}$ and S as follows.

$$S = \begin{pmatrix} 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \end{pmatrix} \tag{8}$$

After $\mathbf{p}$ is announced to all bidders by the auctioneer, they submit their $M^i(\mathbf{p})$ respectively:

$M^a(\mathbf{p})$={3, 4}
$M^b(\mathbf{p})$={1, 2, 3}
$M^c(\mathbf{p})$={2, 3, 4}
$M^d(\mathbf{p})$={1, 3}
$M^e(\mathbf{p})$={1, 3, 4}.

For each $j \in M^i(\mathbf{p})$, let $S_{i,j} := p_j$, then the matrix S becomes:

$$S = \begin{pmatrix} 0 & -\infty & -\infty & 1 & 5 \\ 0 & 5 & 4 & 1 & -\infty \\ 0 & 5 & 4 & 1 & 5 \\ 0 & 5 & 4 & 1 & -\infty \\ 0 & 5 & -\infty & 1 & 5 \end{pmatrix} \tag{9}$$

Now $O(\mathbf{p}) = \{1, 2, 3, 4\}$, i.e., all the items except the dummy item are over-demanded. $p_1 < \bar{p}_1$, $p_2 < \bar{p}_2$, $p_3 < \bar{p}_3$ and $p_4 < \bar{p}_4$. We then let $p_1 := p_1 + 1$, $p_2 := p_2 + 1$,

$p_3 := p_3 + 1$, $p_4 := p_4 + 1$, and the price vector is adjusted to $\mathbf{p} = (0, 6, 5, 2, 6)$. The auctioneer announces the new price vector and asks all bidders resubmit their demand sets, which are$M^a(\mathbf{p}) = \{3, 4\}$, $M^b(\mathbf{p}) = \{1, 2, 3\}$, $M^c(\mathbf{p}) = \{3, 4\}$, $M^d(\mathbf{p}) = \{1, 3\}$, and $M^e(\mathbf{p}) = \{3, 4\}$.

Use the demand sets and the new price to update matrix S as follows:

$$S = \begin{pmatrix} 0 & -\infty & -\infty & 2 & 6 \\ 0 & 6 & 5 & 2 & -\infty \\ 0 & 5 & 5 & 2 & 6 \\ 0 & 6 & 4 & 2 & -\infty \\ 0 & 6 & -\infty & 2 & 6 \end{pmatrix} \tag{10}$$

In this case, $O(\mathbf{p}) = \{1, 3, 4\}$, $p_1 = \bar{p}_1$, $p_2 < \bar{p}_2$, $p_3 < \bar{p}_3$ and $p_4 < \bar{p}_4$. Let $p_2 := p_2 + 1$, $p_3 := p_3 + 1$ and $p_4 := p_4 + 1$. The auctioneer announces the new price vector $\mathbf{p} = (0, 6, 6, 3, 7)$ and requests the bidders to report their new demands. Assume the new demand sets are:

$M^a(\mathbf{p}) = \{3\}$,
$M^b(\mathbf{p}) = \{1, 2, 3\}$,
$M^c(\mathbf{p}) = \{3\}$,
$M^d(\mathbf{p}) = \{1\}$ and
$M^e(\mathbf{p}) = \{4\}$.

Then the matrix S becomes

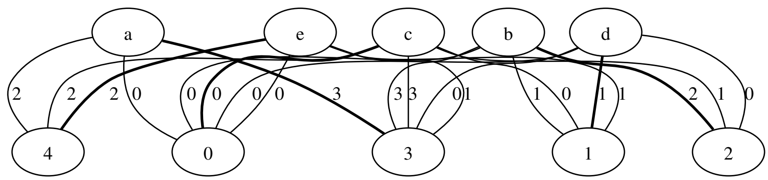
$$S = \begin{pmatrix} 0 & -\infty & -\infty & 3 & 7 \\ 0 & 6 & 6 & 3 & -\infty \\ 0 & 5 & 5 & 3 & 7 \\ 0 & 6 & 4 & 3 & -\infty \\ 0 & 6 & -\infty & 3 & 7 \end{pmatrix} \tag{11}$$

Increase one unit of the price of item 3 because $O(\mathbf{p}) = \{1, 3\}$ and $p_3 < \bar{p}_3$. After the auctioneer announces $\mathbf{p} = (0, 6, 6, 4, 7)$, the bidders report their demands again: $M^a(\mathbf{p}) = \{3\}$, $M^b(\mathbf{p}) = \{1, 2, 3\}$, $M^c(\mathbf{p}) = \{3\}$, $M^d(\mathbf{p}) = \{1\}$ and $M^e(\mathbf{p}) = \{4\}$. The matrix S can be rebuilt according to the current demands and price vector $\mathbf{p}$.

$$S = \begin{pmatrix} 0 & -\infty & -\infty & 4 & 7 \\ 0 & 6 & 6 & 4 & -\infty \\ 0 & 5 & 5 & 4 & 7 \\ 0 & 6 & 4 & 3 & -\infty \\ 0 & 6 & -\infty & 4 & 7 \end{pmatrix} \tag{12}$$

At this time, we find that only item 1 and item 3 are over-demanded, but their prices have both reached their upper bound. The auction procedure stops because $p_j = \bar{p}_j$ for all $j \in O(\mathbf{p})$.

According to matrix S, we build a weighted graph $G = (N \cup X, E, W)$, where $S_{i,j} \in E$ iff $S_{i,j} \neq -\infty$ and $W(e_{i,j}) = S_{i,j}$ for each $i \in N, j \in X$. It is easy to verify that $\Omega = \{(a, 3), (b, 2), (c, 0), (d, 1), (e, 4)\}$ is a maximum weight matching, which determines an allocation $\pi^* = (3, 2, 0, 1, 4)$. The following picture shows the weighted graph. Bold lines represent the maximum weight matching.



The rationing system is then $R^* = (R^{*a}, R^{*b}, R^{*c}, R^{*d}, R^{*e})$ where
$R^{*a} = (1, 1, 1, 1, 0)$
$R^{*b} = (1, 0, 1, 0, 1)$
$R^{*c} = (1, 1, 1, 0, 0)$
$R^{*d} = (1, 1, 1, 1, 1)$
$R^{*e} = (1, 0, 1, 0, 1)$.
The equilibrium price is $P^* = (0, 6, 6, 4, 7)$. The equilibrium implements a market value at 8.

We would like to remark that the constrained Walrasian equilibrium allocation under Talman and Yang's definition is $(0, 3, 2, 1, 4)$ at price $(0, 5, 4, 2, 5)$. The associated market value is 7, which is lower than the efficient competitive equilibrium.

5 Conclusion and Related Work

In this paper, we have introduced a concept of competitive equilibrium by weakening the concept of constrained Walrasian equilibrium. We have devised an dynamic auction procedure and prove that it can generates an efficient competitive equilibrium for any economy for selling in dividable items with price rigidities.

For the purpose of controlling price macroscopically, preventing speculation or protecting the profits of low-incoming buyers, price rigidity is widely adopted to restrict the price of each item to an interval. The phenomenon of price rigidity, i.e., the persistence of price at which supply and demand are not equal, is frequently observed, and plays an important role in some macro-economic models [2]. After investigating the ability of nominal price rigidity, a dynamic general equilibrium model is constructed by [6] with the introduction of monopolistic competition and nominal price rigidity in a standard real business cycle model, allowing for an endogenous money supply rule. From the aspect of banking industry, the price rigidity is significantly greater in markets characterised by higher levels of concentration [7].

Ausubel proposed a dynamic auction procedure for auctioning multiple heterogeneous commodities, and this auction yields a Walrasian equilibrium price and an efficient allocation without considering price rigidities [8]. The Vickrey and Groves-Clarke auctions can be generalised to attain efficiency when there are common values, if each buyers' information can be represented as a one-dimensional signal. Also, when a buyer's information is multidimensional, no auction is generally efficient [9].

Subsequently, Talman and Yang proposed a dynamic auction for differentiated items under price rigidities and by which yielded a constrained Walrasian equilibrium in finite steps [4]. As can be seen from the procedure of dynamic auction, a group of constrained

Walrasian equilibria taking the form (p^*, π^*, R^*) can be generated. It is obviously that each of their social efficiencies at certain price vector $\mathbf{p}$ can be computed, but not all of them have the same efficiency. So, these constrained Walrasian equilibria are not efficient.

Motivated by the difficulty to achieve the social efficiency under price rigidities, the dynamic auction, suggested by this paper, invite bidders to present their demand set for all items so as to promote the possibility to be allotted an item and drive price ascending under over demands. The efficient competitive equilibrium can be found by the dynamic auction procedure in a finite number of step. Also, this dynamic auction procedure is useful to discover the social revenue of auctioneer. For the further research, we will devote ourself to analysis, present and value the relations among different items, such as substitute relation and complement relation, because these relations effect bidders' strength of demands and the distance of price ascending.

References

1. Sun, N., Yang, Z.: A double-track adjustment process for discrete markets with substitutes and complements. Econometrica 77(3), 933–952 (2009)
2. Dreze, J.H.: Existence of an exchange equilibrium under price rigidities. International Economic Review 16(2), 301–320 (1975)
3. Herings, J.J., Talman, D., Yang, Z.: The computation of a continuum of constrained equilibria. Mathematics of Operations Research 21(3) (1996)
4. Talman, D., Yang, Z.: A dynamic auction for differentiated items under price rigidities. Economics Letters 99(2), 278–281 (2008)
5. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency, vol. A,B,C. Springer, Heidelberg (2004)
6. Yun, T.: Nominal price rigidity, money supply endogeneity, and business cycles. Journal of Monetary Economics 37(2-3), 345–370 (1996)
7. Hannan, T.H., Berger, A.N.: The rigidity of prices: Evidence from the banking industry. American Economic Review 81(4), 938–945 (1991)
8. Ausubel, L.M.: An efficient dynamic auction for heterogeneous commodities. The American Economic Review 96(3), 602–629 (2006)
9. Dasgupta, P., Maskin, E.: Efficient auctions. The Quarterly Journal of Economics 115(2), 341–388 (2000)