

A HIERARCHICAL APPROACH TO JUDGMENT AGGREGATION WITH ABSTENTIONS

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Judgment aggregation deals with the problem of how collective judgments on logically connected propositions can be formed based on individual judgments on the same propositions. The existing literature on judgment aggregation mainly focuses on the anonymity condition requiring that individual judgments be treated equally. However, in many real-world situations, a group making collective judgments may assign individual members or subgroups different priorities to determine the collective judgment. Based on this consideration, this article relaxes the anonymity condition by giving a hierarchy over individuals so as to investigate how the judgment from each individual affects the group judgment in such a hierarchical environment. Moreover, we assume that an individual can abstain from voting on a proposition and the collective judgment on a proposition can be undetermined, which means that we do not require completeness at both individual and collective levels. In this new setting, we first identify an impossibility result and explore a set of plausible conditions in terms of abstentions. Secondly, we develop an aggregation rule based on the hierarchy of individuals and show that the aggregation rule satisfies those plausible conditions. The computational complexity of this rule is also investigated. Finally, we show that the proposed rule is (weakly) oligarchic over a subset of agenda. This is by no means a negative result. In fact, our result reveals that with abstentions, oligarchic aggregation is not necessary to be a single-level determination but can be a multiple-level collective decision making, which partially explains its ubiquity in the real world.

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1. INTRODUCTION

Judgment aggregation is an interdisciplinary research topic in economics, philosophy, political science, law, and recently in computer science (Wilson 1975; Mongin 1995; Konieczny and Pérez 2002; Pigozzi 2006; Dietrich and List 2007a; Brandt et al. 2013). It deals with the problem of how a set of group judgments on certain issues, represented by logical propositions, can be formed based on individuals' judgments on the same issues. Although most of voting rules for social choice, such as majority rules, unanimity rules or dictatorships, are applicable to judgment aggregation, their behavior can be significantly different because of possible logical links among the propositions on which a collective decision has to be made. A well-known example is the so-called doctrinal paradox (Kornhauser and Sage 1993; List 2012), which shows that the majority rule fails to guarantee consistent group judgments.

Suppose a court consisting of three judges has to reach a verdict in a breach-of-contract case. There are three propositions on which the court is required to make judgments:

p : The defendant was contractually obliged not to do a particular action.

q : The defendant did that action.

r : The defendant is liable for a breach of contract.

According to the legal doctrine, propositions p and q are jointly necessary and sufficient for proposition r , that is, $p \wedge q \leftrightarrow r$. Now, the three judgments on the propositions are shown in

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TABLE 1. A Doctrinal Paradox.

	p	q	r
Judge 1	Yes	Yes	Yes
Judge 2	Yes	No	No
Judge 3	No	Yes	No
Majority	Yes	Yes	No

Table 1. If the three judges take a majority vote on proposition r , the outcome is its rejection: a *not liable* verdict. However, if they take majority votes on each of p and q instead, then p and q are accepted; thus, by the legal doctrine, r should be accepted as well: a *liable* verdict. Although each judge holds consistent individual judgments on the three propositions, there are majorities for p , q , and $\neg r$, a logically inconsistent set of propositions with respect to the constraint $p \wedge q \leftrightarrow r$.

In aggregating individual judgments, there are two natural demands: one is that a group should be responsive to the views of members on each of judgments involved and the other is that a group should reach a set of collective judgments that is itself consistent. The paradox shows that the two demands are sometimes in conflict; thus, a group that tries to aggregate judgments faces a dilemma. More significantly, List and Pettit (2002) showed an impossibility result, similar to Arrow's (1950) impossibility theorem that no aggregation rule can generate consistent collective judgments if we require an aggregation rule to satisfy a set of *plausible* conditions. However, such an impossibility result did not discourage the investigation of judgment aggregation. None of the conditions on either aggregation rules or decision problems is indefectible.

Among all the conditions that lead to impossibility results on judgment aggregation, *completeness*, as one of the rationality requirements, has received criticism of being overly demanding in many real-world situations, where an individual may abstain from voting on a decision issue and a group judgment on some issue may be undetermined. In fact, if we give up completeness, we are able to circumvent impossibility (List and Pettit 2002; Gärdenfors 2006; Dietrich and List 2007b; 2008a; Dokow and Holzman 2010). In particular, Gärdenfors (2006) proved a representation theorem for judgment aggregation without completeness, which shows that under certain fairly natural conditions, the only possible aggregation rules are oligarchic. Dietrich and List (2008a) strengthened Gärdenfors's results and showed that by giving up completeness in favor of deductive closure, oligarchies instead of dictatorships are obtained. We will continue this line of research and investigate how judgments from each individual affect the group judgments in such an oligarchic environment.

Moreover, in many real-world decision-making settings, when a group forms collective judgments, some group members or subgroups may have priority to decide certain propositions. For instance, legislatures or expert panels may assign specialist members such priority so as to rely on their expertise; when some propositions concern group members' private spheres, they may also be assigned the rights to be decisive on those propositions (Dietrich and List 2008b). However, as pointed out by Dietrich and List (2008b), such kind of priority has been rarely investigated in the current literature on judgment aggregation. In particular, they proposed a generalization of Sen's (1970) *liberal paradox*. Under plausible conditions, the assignment of rights to two or more individuals or subgroups is inconsistent with the unanimity principle requiring unanimously accepted propositions should be collectively accepted. Following their work, most of the existing related literature is concerned with liberal (im)possibility results (Van Hees 1999; Dietrich and List 2008b; Patty and Penn

2014). Yet there are seldom specific aggregation rules designed for a group with priority over its members to generate consistent collective judgments.

To fill this gap, this article provides a hierarchical approach to deal with the priority over individuals and develops a plausible aggregation rule based on individuals' hierarchy so as to investigate how individual judgments affect the collective judgment in such a hierarchical environment. Specifically, our contributions can be summarized as follows:

- First, we propose a more realistic framework for judgment aggregation by allowing an individual or a group to abstain from voting on a proposition and explore a set of intuitively plausible conditions in terms of abstentions. Meanwhile, as a by-product, we identify an impossibility result in this new setting.
- Second, we design a feasible rule for judgment aggregation with abstentions based on voter's hierarchy and verify that it satisfies all the plausible conditions. We also show that its complexity to compute the collective result of a given profile of individual judgments is in polynomial time.
- Third, we verify that the proposed rule is neither dictatorial nor oligarchic over the whole agenda and reveal that with abstentions, oligarchic aggregation is no longer a single-level determination but can also be a multiple-level collective decision making.

The rest of this article is structured as follows: Section 2 introduces a logical model for judgment aggregation with abstentions. Section 3 identifies an impossibility result in this new setting and investigates the plausible conditions for aggregation rules in terms of abstentions. Section 4 proposes a feasible judgment aggregation procedure to deal with judgment aggregation with abstentions based on voters' hierarchy and investigates its properties as well as computational complexity. Section 5 explores the oligarchic property of the proposed rule and shows that with abstentions oligarchic aggregation is no longer a single-level determination but can also be a multiple-level collective decision making. Finally, we conclude this article with a discussion of related work, potential applications, and further work.

2. THE MODEL OF JUDGMENT AGGREGATION WITH ABSTENTIONS

We consider a finite set of individuals $N = \{1, 2, \dots, n\}$ with $|N| \geq 2$. They face a decision problem that requires collective judgments on propositions. Propositions are represented by a logical language \mathcal{L} with a set Φ_0 of propositional variables and standard logical connectives $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$. Following List and Petit (2002, 2004), we assume that the underlying logic is the classical propositional logic with standard syntax and semantics. The set of literals, denoted by \mathcal{P} , consists of either propositional variables or their negations, i.e., $\mathcal{P} = \{p, \neg p \mid p \in \Phi_0\}$.

The set of propositions on which judgments are to be made is called the *agenda*. Formally, the agenda is a finite non-empty subset $X \subseteq \mathcal{L}$ closed under negation (i.e., if $\varphi \in X$, then $\neg\varphi \in X$) and under propositional variables (i.e., for all $\varphi \in \mathcal{L}$, if $\varphi \in X$, then for all $p \in \Phi_0$ occur in φ , $p \in X$). Similar to (Dietrich and List 2008a), we assume that double negations in the agenda cancel each other. That is, $X = \{\varphi, \neg\varphi : \varphi \in X^*\}$, where $X^* \subseteq \mathcal{L}$ is a set of unnegated propositions. Let $X_0 = X \cap \mathcal{P}$ be the set of literals included in the agenda. Consider the doctrinal paradox in Section 1. The agenda is

$$\{p, q, p \wedge q, \neg p, \neg q, \neg(p \wedge q)\}.$$

The set of literals in the agenda is

$$\{p, q, \neg p, \neg q\}.$$

Note that for the sake of readability, we replace r by $p \wedge q$ as they are logically equivalent. We call a set $Y \subseteq \mathcal{L}$ *minimally inconsistent* if it is inconsistent and every proper subset of Y is consistent. For instance, $\{p, q, \neg(p \wedge q)\}$ is a minimally inconsistent set, while $\{p, \neg q, p \wedge q\}$ is not, because its proper subset $\{\neg q, p \wedge q\}$ is inconsistent. The agenda X is *non-simple* if it has a minimal inconsistent subset Y such that $|Y| \geq 3$. For instance, the agenda in the doctrinal paradox is non-simple, because it includes a minimal inconsistent subset $\{p, q, \neg(p \wedge q)\}$.

We represent each *individual judgment set* as a subset of the agenda, which indicates all the propositions that this individual accepts or believes to be true. Formally, individual i 's judgment set, denoted by Φ_i , is a subset of X , i.e., $\Phi_i \subseteq X$. We assume that each individual judgment set Φ_i satisfies the following conditions:

- *Consistence*: All its members can be simultaneously true, i.e., $\Phi_i \not\models \perp$.
- *Deductive closure*: For every $\varphi \in X$, if $\Phi_i \models \varphi$, then $\varphi \in \Phi_i$.

As we have mentioned in Section 1, we do not require Φ_i to be complete, i.e., for every pair $\varphi, \neg\varphi \in X$, either $\varphi \in \Phi_i$ or $\neg\varphi \in \Phi_i$. Then for each proposition $\varphi \in X$, it may happen that $\varphi \notin \Phi_i$ and $\neg\varphi \notin \Phi_i$. In this case, we say that individual i abstains from making a judgment on φ . Given a proposition $\varphi \in X$, individual i 's judgment on this proposition may be acceptance (i.e., $\varphi \in \Phi_i$), rejection (i.e., $\neg\varphi \in \Phi_i$), and abstention, denoted as $+$, $-$ and $\#$, respectively. With abstentions, if an individual makes her judgments over φ and ψ , respectively, then her judgments on their compound formulas must be consistent with the judgments of φ and ψ . The composition of two judgments with respect to \rightarrow is depicted in Table 2. For instance, if an individual abstains from voting on φ (i.e., $\#$) and rejects ψ (i.e., $-$), then she should abstain from voting on $\varphi \rightarrow \psi$ (i.e., $\#$). Otherwise, her individual judgment set would be inconsistent. Given each individual judgment set Φ_i , the vector $\langle \Phi_i \rangle_{i \in N}$ is called a *profile of individual judgment sets*. For instance, in the doctrinal paradox, the profile is

$$\langle \{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}, \{\neg p, q, \neg(p \wedge q)\} \rangle.$$

Finally, a *judgment aggregation rule* is a function F that assigns to each profile $\langle \Phi_i \rangle_{i \in N}$ a single collective judgment set $\Phi \subseteq X$, where $\varphi \in \Phi$ means that the group as a whole accepts φ . The set of all admissible profiles is called the *domain* of F , denoted by $Dom(F)$. Note that we do not require the collective judgments to be complete, which allows that a group may abstain from voting on certain propositions. In the following, we will impose plausible conditions on aggregation rules. A standard example of aggregation rules is *the majority rule*, as introduced in the doctrinal paradox, where each proposition is collectively

TABLE 2. The Compositions for \rightarrow .

		ψ		
		$+$	$\#$	$-$
$\varphi \rightarrow \psi$	$+$	$+$	$\#$	$-$
	$\#$	$+$	$\#$	$\#$
	$-$	$+$	$+$	$+$

accepted if and only if the number of individuals accepted this proposition exceeds the half, i.e.,

$$F(\langle \Phi_i \rangle_{i \in N}) = \{\varphi \in X : |\{i \in N : \varphi \in \Phi_i\}| > n/2\}.$$

3. CONDITIONS ON AGGREGATION RULES

In this section, we investigate plausible conditions for aggregation rules in terms of abstentions. We begin with a simple impossibility result in this new context.

3.1. Impossibility Result

Similar to preference aggregation (Gaertner 2009), a judgment aggregation rule is supposed to satisfy a set of plausible conditions. Let F be a judgment aggregation rule. We consider the following three conditions:

- **Universal domain (UD).** The domain of F includes all possible profiles of individual judgment sets.
- **Non-dictatorship (ND).** *There is no $x \in N$ such that for all $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$, $F(\langle \Phi_i \rangle_{i \in N}) = \Phi_x$.* This is a basic democratic requirement: no single individual should always determine the collective judgment set.
- **Unanimity (U) with abstentions.** *For every $\varphi \in X$, if there is some $V \subseteq N$ such that $V \neq \emptyset$, for all $i \in V$, $\varphi \in \Phi_i$ and for all $j \in N \setminus V$, $\varphi \notin \Phi_j$, then $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$.* Intuitively, if a non-empty set of individuals agrees on accepting a proposition φ , while all the others abstain from voting on φ , then this condition requires that $F(\langle \Phi_i \rangle_{i \in N})$ should accept φ as well.

It is worth noting that unanimity with abstentions is the counterpart of condition *be unanimous with abstentions* in Andréka et al. (2002), and *Pareto optimality* in Gärdenfors (2006) (also called *unanimity principle* in Dietrich and List [2008b], *Paretian condition* in Dokow and Holzman [2010]) can be derived from it.

Proposition 1. *If F satisfies unanimity with abstentions, then F is Pareto optimal, i.e., for every $\varphi \in X$, if $\varphi \in \Phi_i$ for every $i \in N$, then $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$.*

The next proposition says that ND can be derived from unanimity with abstentions.

Proposition 2. *Every judgment aggregation rule satisfying unanimity with abstentions is non-dictatorial.*

Proof. Assume that F is dictatorial in some individual $x \in N$, then $N \setminus \{x\} \neq \emptyset$. Take $\varphi \in X$ and define $\varphi \notin \Phi_x$ and $\varphi \in \Phi_i$ for every $i \in N \setminus \{x\}$. By unanimity with abstentions, $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$, then $F(\langle \Phi_i \rangle_{i \in N}) \neq \Phi_x$, contradiction. ■

On the other hand, the following example shows that this condition fails to guarantee consistent judgment sets.

Example 1. Suppose Ann, Bill, and Tom have to make group judgments on three logically connected propositions as follows:

p : There is the elixir of life.

q : Humans can be immortal.

$p \rightarrow q$: If there is the elixir of life, then humans can be immortal.

TABLE 3. Individual Judgments in Example 1.

	p	q	$p \rightarrow q$
Ann	+	#	#
Bill	#	-	#
Tom	#	#	+

Their individual judgments are shown in Table 3.

According to the unanimity with abstentions, the collective judgment set is $\{p, p \rightarrow q, \neg q\}$, which is inconsistent. In fact, the following result shows that this is not an individual case.

Theorem 1. If and only if the agenda is non-simple, no aggregation rule generates consistent collective judgment sets and satisfies universal domain and unanimity with abstentions.

Proof. First, assume the agenda is non-simple. Suppose not for a contradiction that there is an aggregation rule F satisfying universal domain and unanimity with abstentions. We next show that F generates an inconsistent collective judgment set on some profile. By assumption that the agenda is non-simple, then there is a minimally inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. Let α, β , and γ be the three distinct propositions in Y . Consider a profile $\langle \Phi_i \rangle_{i \in N}$ such that $\Phi_1 = Y \setminus \{\beta\}$, $\Phi_2 = Y \setminus \{\alpha\}$ and $\Phi_i = Y \setminus \{\gamma\}$ for any $i \in N \setminus \{1, 2\}$. Then $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$. And by unanimity of abstention, $Y \subseteq F(\langle \Phi_i \rangle_{i \in N})$; thus, $F(\langle \Phi_i \rangle_{i \in N})$ is inconsistent.

Now, assume the agenda is simple. Then there is no minimally inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. Let F be the aggregation rule with universal domain as follows: for all $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$ and for all $\varphi \in X$,

$$\varphi \in F(\langle \Phi_i \rangle_{i \in N}) \text{ iff } \forall i \in N (\neg \varphi \notin \Phi_i) \text{ and } \exists x \in N (\varphi \in \Phi_x)$$

We next prove that F satisfies all requirements.

To show that F satisfies unanimity with abstentions, we assume for any $\varphi \in X$, there is some $V \subseteq N$ such that $V \neq \emptyset$, $\forall i \in V. \varphi \in \Phi_i$ and $\forall j \in N \setminus V. \varphi \# \Phi_j$, then $\neg \varphi \notin \Phi_i$ for any $i \in N$. And by $V \neq \emptyset$, there is $x \in V$ such that $\varphi \in \Phi_x$. Thus, it follows from the definition of F that $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$.

Finally, we consider any profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$ and show that $F(\langle \Phi_i \rangle_{i \in N})$ is consistent. Suppose for a contradiction that $F(\langle \Phi_i \rangle_{i \in N})$ is inconsistent for some profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$. As $F(\langle \Phi_i \rangle_{i \in N}) \subseteq X$ and by assumption that X is simple, then there is some $p \in X_0$ such that $p \in F(\langle \Phi_i \rangle_{i \in N})$ and $\neg p \in F(\langle \Phi_i \rangle_{i \in N})$. According to the definition of F , $(\forall i \in N (\neg p \notin \Phi_i))$ and $\exists x \in N (p \in \Phi_x)$, and $(\forall i \in N (p \notin \Phi_i))$ and $\exists y \in N (\neg p \in \Phi_y)$, contradiction. Hence, $F(\langle \Phi_i \rangle_{i \in N})$ is consistent for any profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$. ■

On the one hand, this impossibility result indicates that in decision process, when the agenda is logically connected, ignoring abstentions would lead to inconsistent collective results. In fact, unanimity with abstentions means that an individual has the right to determine the collective judgment on certain propositions whenever all the others abstain from voting on that proposition. Thus, to some extent, this impossibility result is a version of Dietrich

and List's liberal impossibility result in terms of abstentions. On the other hand, this result provides a characterization theorem for the class of non-simple agendas. It shows that for the class of non-simple agendas, a combination of conditions universal domain, collective consistence, and unanimity with abstentions leads to an empty class of aggregation rules. And it also fully characterizes those agendas for which this is the case and, by implication, those for which it is not.

3.2. Conditions on Aggregation Rules

In this subsection, we investigate desirable conditions for aggregation rules under which the impossibility does not arise. The first condition is a variant of unanimity with abstention by restricting propositions to the set of literals.

- **Literal unanimity (LU) with abstentions.** *For every $\alpha \in \mathcal{P}$, if there is some $V \subseteq N$ such that $V \neq \emptyset$, for all $i \in V$. $\alpha \in \Phi_i$ and for all $j \in N \setminus V$. $\alpha \notin \Phi_j$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$.*

This condition is a restriction of unanimity with abstentions. As we will see in the following example, it plays a crucial role in extending the agenda set from a set of literals to a set of logically interconnected formulas without generating inconsistent aggregate results. It is worth noting that it is neither a restriction nor an extension of Pareto optimality and ND can still be derived from LU.

Proposition 3. Every judgment aggregation rule satisfying LU with abstentions is non-dictatorial.

The following independence condition requires that the group judgment on each literal should depend only on individual judgments on that literal, which is a counterpart of Arrow's (1953) "independence of irrelevant alternative." With abstentions, this condition has two different versions:

- **Strong literal independence (LI^s).** *For every $\alpha \in \mathcal{P}$ and every profiles $\langle \Phi_i \rangle_{i \in N}$, $\langle \Phi'_i \rangle_{i \in N} \in \text{Dom}(F)$, if $\alpha \in \Phi_i \leftrightarrow \alpha \in \Phi'_i$ for every $i \in N$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \alpha \in F(\langle \Phi'_i \rangle_{i \in N})$.*
- **Weak literal independence (LI).** *For every $\alpha \in \mathcal{P}$ and every profiles $\langle \Phi_i \rangle_{i \in N}$, $\langle \Phi'_i \rangle_{i \in N} \in \text{Dom}(F)$, if $\alpha \in \Phi_i \leftrightarrow \alpha \in \Phi'_i$ and $\neg\alpha \in \Phi_i \leftrightarrow \neg\alpha \in \Phi'_i$ for every $i \in N$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \alpha \in F(\langle \Phi'_i \rangle_{i \in N})$.*

Note that these two versions are the same if we assume the completeness. With abstentions, these two versions become different. The strong version is intuitively too strong to be acceptable, because it requires that even if all judges who abstain from voting on α turn to rejecting α , no matter how big portion of these judges is, the group judgment on α should be the same so long as the same set of judges accept α . The weak version solves this problem. Similar conditions have been also discussed in the literature. It is not hard to see that the *strong LI* condition strengthens the *independence of irrelevant propositional alternatives condition* in Mongin (2008) to the set of literals under the provision of abstentions, while the weak independence condition amounts to reserving the *independent of irrelevant alternatives condition* in Dokow and Holzman (2010) to the set of literals.

The following two conditions are two versions of the counterpart of the *neutrality* condition (List and Polak 2010), which requires that all literals should be treated in an even-handed way.

- **Strong literal neutrality.** For every $\alpha, \beta \in \mathcal{P}$ and every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$, if $\alpha \in \Phi_i \leftrightarrow \beta \in \Phi_i$ for every $i \in N$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \beta \in F(\langle \Phi_i \rangle_{i \in N})$.
- **Weak literal neutrality.** For every $\alpha, \beta \in \mathcal{P}$ and every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$, if $\alpha \in \Phi_i \leftrightarrow \beta \in \Phi_i$ and $\neg\alpha \in \Phi_i \leftrightarrow \neg\beta \in \Phi_i$ for every $i \in N$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \beta \in F(\langle \Phi_i \rangle_{i \in N})$.

Similarly, these two versions are the same if we assume individual completeness. The strong version is intuitively too strong to be acceptable, because it requires that even if all judges who abstain from voting on α reject β , no matter how big portion of these judges is, the group judgment on α and β should be the same so long as the set of judges accepting α and the set of judges accepting β are the same. The weak version avoid this phenomenon.

The last condition is the counterpart of *systematicity*, introduced by List and Pettit (2002), which combines independency and neutrality.

- **Strong literal systematicity (LS^s).** For every $\alpha, \beta \in \mathcal{P}$ and every profiles $\langle \Phi_i \rangle_{i \in N}, \langle \Phi'_i \rangle_{i \in N} \in \text{Dom}(F)$, if for every $i \in N$, $\alpha \in \Phi_i \leftrightarrow \beta \in \Phi'_i$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \beta \in F(\langle \Phi'_i \rangle_{i \in N})$.
- **Weak literal systematicity (LS).** For every $\alpha, \beta \in \mathcal{P}$ and every profiles $\langle \Phi_i \rangle_{i \in N}, \langle \Phi'_i \rangle_{i \in N} \in \text{Dom}(F)$, if for every $i \in N$, $\alpha \in \Phi_i \leftrightarrow \beta \in \Phi'_i$ and $\neg\alpha \in \Phi_i \leftrightarrow \neg\beta \in \Phi'_i$, then $\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \leftrightarrow \beta \in F(\langle \Phi'_i \rangle_{i \in N})$.

The reason why we reserve independence, neutrality, and systematicity to literals alone is based on the consideration that the problem of the doctrinal paradox in Section 1 comes from the requirement that the majority rule treats the compound formulas and propositional variables independently. Indeed, *the principle of compositionality*, a fundamental presupposition of the semantics in most contemporary logics, denotes that the propositional variables are more primary than the compound formulas, because the truth of the latter is determined by the truth of the former. For instance, in the doctrinal paradox, the truth of the conjunctive formula $p \wedge q$ is determined by its constituents p and q . In this sense, we may say the judgments on p and q are the reasons to accept $p \wedge q$ or not, while the reason for whether p or q is accepted or not is beyond the expressivity of propositional logic.

Therefore, we take a reason-based perspective and apply the aggregation rule only to primary data whose reasons are beyond the expressivity power of the underlying logics, then use them to generate complex formulas within the underlying logic (Mongin 2008; Nehring and Puppe 2008). Given abstentions, it is the literals instead of propositional variables that are primary data. Without completeness, we can not derive that p is rejected from that p is not accepted. It might be possible that p is undetermined (neither accepted nor rejected). Therefore, we reserve independence, neutrality, and systematicity to literals instead of propositional variables. On the one hand, this makes them more acceptable. For instance, one criticizes systematicity (the independent part) being used for $p \vee q$, where p denotes “The government can afford a budget deficit” and q “Forbidding smoking should be legalized” on the ground that there are two propositions involved and that the society should know how each individual feels about both propositions and not just about their disjunction. There is no similar objection arising when systematicity applies to either p or q (Mongin 2008). On the other hand, this provides a plausible solution for the paradox shown in Section 1. Let us apply the majority rule to literals and calculate

p and q in the group judgment set, then use them to generate $p \wedge q$ in the group judgment set. Thereby, the group judgment set is $\{p, q, r\}$ which is logically consistent with the legal doctrine.

4. AGGREGATION RULE UNDER VOTERS' HIERARCHY

In this section, we propose a feasible rule for hierarchical groups based on the lexicographic rule in Andréka et al. (2002) and show that the proposed rule satisfies the plausible conditions in Section 3.2. We also investigate the computational complexity of this rule.

4.1. The Literal-Based Lexicographic Rule

We first treat individuals' priorities as a hierarchy among individuals. In the real-world, we can easily see such a hierarchy, such as the management structure of an enterprise, a democratic political regime, or a community organization. Members in different ranks may play different roles in collective decision making. Formally,

Definition 1. A *hierarchy* over the set N of individuals is a strict partial order $<\subseteq N \times N$, i.e., $<$ is transitive and asymmetric.

Intuitively, for two individuals $a, b \in N$, $a < b$ means that b is more important than a . Note that since N is finite, so there is no infinite ascending sequence $i_1 < i_2 < i_3 < \dots$, where $i_n \in N$, which means all hierarchical chains of N must be *up-bounded* with at least one top leader. In this sense, we call $(N, <)$ is *well prioritized*.

An aggregation rule determines which propositions are collectively accepted and which ones are collectively rejected. Recall that $X_0 = X \cap \mathcal{P}$ is the set of literals in the agenda. We define an aggregation procedure F that $\alpha \in X_0$ is collectively accepted, denoted by $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$, as follows:

Definition 2. For every $\alpha \in X_0$,

$$\alpha \in F(\langle \Phi_i \rangle_{i \in N}) \text{ iff } \forall_{i \in N} (\neg \alpha \notin \Phi_i \vee \exists_{j \in N} (i < j \wedge \alpha \in \Phi_j)) \text{ and } \exists_{k \in N} (\alpha \in \Phi_k) \quad (1)$$

Intuitively, this aggregate procedure says that a literal α is accepted by a group if the following two conditions are both satisfied.

- For any individual, if she rejects α , then there is an individual with higher hierarchy accepting α , and
- there is at least one individual accepting α .

We denote the set of collectively accepted literals by $F(\langle \Phi_i \rangle_{i \in N})_0$. Based on this, we next define that any $\varphi \in X$ is collectively accepted as follows:

Definition 3. For any $\varphi \in X$,

$$\varphi \in F(\langle \Phi_i \rangle_{i \in N}) \text{ iff } F(\langle \Phi_i \rangle_{i \in N})_0 \models \varphi \quad (2)$$

This definition says that a proposition φ in the agenda X is collectively accepted if it is a logical consequence of the collectively accepted literals.

Similarly, a proposition $\varphi \in X$ is *collectively undetermined* if neither itself nor its negation is collectively accepted. That is,

$$\varphi \# F(\langle \Phi_i \rangle_{i \in N}) \text{ iff } \varphi \notin F(\langle \Phi_i \rangle_{i \in N}) \text{ and } \neg \varphi \notin F(\langle \Phi_i \rangle_{i \in N}). \quad (3)$$

We call previously defined judgment aggregation rule F the *literal-based lexicographic (judgment aggregation) rule* as we just apply the lexicographic rule to the subset of literals in the agenda. Similar to the lexicographic rule, the higher an individual is in the hierarchy, the more important she is.

To demonstrate how this aggregation rule works, let us consider the following example obtained by a slight change of Example 1.

Example 2. Suppose Ann, Bill, and Tom have to make group judgments on three logically connected propositions as follows:

- p : There is the elixir of life.
- q : Humans can be immortal.
- $p \rightarrow q$: If there is the elixir of life, then human can be immortal.

Ann agrees p , but she is not sure whether q and $p \rightarrow q$ are true; Bill rejects q and abstains from voting on p and $p \rightarrow q$; Tom accepts q and $p \rightarrow q$ and leaves p undetermined. Their individual judgments are shown in Table 4. The hierarchy among them is $\text{Ann} < \text{Bill}$ and $\text{Tom} < \text{Bill}$ as illustrated in Figure 1, where individuals with the highest priority are written at the top of the diagram.

We next apply the literal-based aggregation rule to generate the collective judgment set. The model of this aggregation situation is as follows:

- $N = \{\text{Ann}, \text{Bill}, \text{Tom}\}$ with $\text{Ann} < \text{Bill}$ and $\text{Tom} < \text{Bill}$;
- $X = \{p, q, p \rightarrow q, \neg p, \neg q, \neg(p \rightarrow q)\}$ and $X_0 = \{p, q, \neg p, \neg q\}$.

The individual judgment sets for Ann, Bill, and Tom are as follows:

- $\Phi_{\text{Ann}} = \{p\}$, i.e., $p \in \Phi_{\text{Ann}}, q \# \Phi_{\text{Ann}}, p \rightarrow q \# \Phi_{\text{Ann}}$;
- $\Phi_{\text{Bill}} = \{\neg q\}$, i.e., $p \# \Phi_{\text{Bill}}, \neg q \in \Phi_{\text{Bill}}, p \rightarrow q \# \Phi_{\text{Bill}}$;
- $\Phi_{\text{Tom}} = \{q, p \rightarrow q\}$, i.e., $p \# \Phi_{\text{Tom}}, q \in \Phi_{\text{Tom}}, p \rightarrow q \in \Phi_{\text{Tom}}$.

TABLE 4. Individual Judgments in Example 2.

	p	q	$p \rightarrow q$
Ann	+	#	#
Bill	#	-	#
Tom	#	+	+

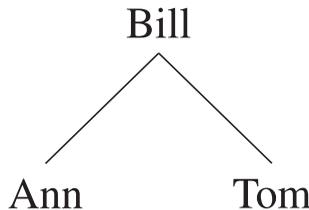


FIGURE 1. The hierarchy in Example 2.

We first calculate the group judgments on the set X_0 of literals by Definition 1.

- The group accepts p , because all of them do not reject p , i.e., $\forall_{i \in N} (\neg p \notin \Phi_i)$ holds, and Ann accepts p , i.e., $\exists_{j \in N} (p \in \Phi_j)$ holds.
- The group rejects q , because Bill with the highest priority rejects q .

Then $F(\langle \Phi_i \rangle_{i \in N})_0 = \{p, \neg q\}$. By Definition 3, the group rejects $(p \rightarrow q)$, because $\{p, \neg q\} \models \neg(p \rightarrow q)$. Thus, the group judgment set is $\{p, \neg q, \neg(p \rightarrow q)\}$ by the literal-based lexicographic rule.

To end this subsection, we would like to mention that according to the literal-based lexicographic rule, the collective result for Example 1 is $\{p, \neg q, \neg(p \rightarrow q)\}$, which becomes consistent. In particular, the literal-based lexicographic rule is flexible to provide a solution to the doctrinal paradox shown in Section 1. We may take all the possible hierarchy among the three agents into consideration. One boss case: let $1 < 2 < 3$ be the hierarchy, then according to this rule the aggregate result is just the individual aggregate set of the first agent $\{p, q, r\}$, which is consistent at the cost that the first agent seems to be the dictator for this profile. Two-boss case: let the hierarchy be $3 < 1$ and $3 < 2$, then they collectively accept p and abstain from voting on q and r , i.e., $\{p\}$. Three-boss (no boss or anonymity) case: p, q , and r are all collectively undetermined.

4.2. Possibility Result

We next show that the proposed aggregation rule satisfies all the desirable conditions in Section 3.2 and thus is a feasible aggregation rule to generate collective judgments.

Theorem 2. The literal-based lexicographic rule F generates consistent and deductively closed collective judgment sets and satisfies conditions UD, LU, and LS.

Proof. Regarding consistence, it suffices to show that for any $\alpha \in \mathcal{P}$, $\alpha \in F(\langle \Phi_i \rangle_{i \in N})_0$ implies $\neg \alpha \notin F(\langle \Phi_i \rangle_{i \in N})_0$. Suppose for a contradiction that for some $\beta \in \mathcal{P}$, $\beta \in F(\langle \Phi_i \rangle_{i \in N})_0$ and $\neg \beta \in F(\langle \Phi_i \rangle_{i \in N})_0$, then (i) $\forall_{i \in N} (\neg \beta \notin \Phi_i \vee \exists_{j \in N} (i < j \wedge \beta \in \Phi_j))$ and $\exists_{k \in N} (\beta \in \Phi_k)$; (ii) $\forall_{i \in N} (\beta \notin \Phi_i \vee \exists_{j \in N} (i < j \wedge \neg \beta \in \Phi_j))$ and $\exists_{k \in N} (\neg \beta \in \Phi_k)$. By (i), (ii) we can obtain an infinite ascending sequence i_1, i_2, i_3, \dots , which is a contradiction with that $(N, <)$ is well prioritized. Then $F(\langle \Phi_i \rangle_{i \in N})_0$ is consistent; thus, by Definition 3, $F(\langle \Phi_i \rangle_{i \in N})$ is consistent as well.

Regarding deductive closure, for any $\varphi \in X$ assume $F(\langle \Phi_i \rangle_{i \in N}) \models \varphi$, then φ is either a literal or a compound formula. If φ is a literal, then $\varphi \in F(\langle \Phi_i \rangle_{i \in N})_0$; thus, $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$. If φ is a compound formula, it is straightforward by Definition 3.

It is easy to see that F satisfies condition UD.

Regarding LU, assume for every $\alpha \in \mathcal{P}$, if there is some $V \subseteq N$ such that $V \neq \emptyset$, $\forall_{i \in V} (\alpha \in \Phi_i)$ and $\forall_{j \in N \setminus V} (\alpha \notin \Phi_j)$, then by Definition 1, $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$.

Regarding LS, given every $\alpha \in \mathcal{P}$, the individuals accepting α and these rejecting α are the same for every two profiles $\langle \Phi_i \rangle_{i \in N}$, $\langle \Phi'_i \rangle_{i \in N}$, then the aggregate results of α according to Definition 1 are the same as well. Yet it is not the case for LS^s. Consider a counterexample: Let $N = \{1, 2, 3\}$ with $1 < 2$, $1 < 3$. For the profile $\langle \Phi_i \rangle_{i \in N}$, where $\alpha \in \Phi_1$, $\alpha \notin \Phi_2$ and $\alpha \notin \Phi_3$, we have $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$ by LU. Let individuals 2 and 3 who abstain from voting on it turn to rejecting α , while individual 1 still accepts α , we obtain a different profile $\langle \Phi'_i \rangle_{i \in N}$, where $\alpha \in \Phi'_1$, $\neg \alpha \in \Phi'_2$ and $\neg \alpha \in \Phi'_3$, then $\alpha \in F(\langle \Phi'_i \rangle_{i \in N})$ by LS^s, but according to the rule, $\alpha \notin F(\langle \Phi'_i \rangle_{i \in N})$. ■

It follows directly that these conditions are coherent, and thus, the following possibility result holds, as we expected.

Corollary 1. There exists an aggregation rule generating consistent collective judgment sets and satisfying conditions UD, LU, and LS.

The last proposition of this section shows under which conditions two rules satisfying LS are identical. We first introduce a helpful notation. For any $i \in N$ and $\varphi, \psi \in X$, the individual i makes the same judgment on φ and ψ is denoted by $\Phi_i|_{\{\varphi\}} \Leftrightarrow \Phi_i|_{\{\psi\}}$, that is, $\Phi_i|_{\{\varphi\}} \Leftrightarrow \Phi_i|_{\{\psi\}}$ iff $(\varphi \in \Phi_i \text{ if } \psi \in \Phi_i)$ and $(\neg\varphi \in \Phi_i \text{ iff } \neg\psi \in \Phi_i)$.

Proposition 4. Let $\{p\} \subseteq X \cap \Phi_0$ and F_1, F_2 be two aggregation rules satisfying LS. If for all profiles of individual judgment sets $\langle \Phi_i \rangle_{i \in N}$, $F_1(\langle \Phi_i \rangle_{i \in N})|_{\{p\}} \Leftrightarrow F_2(\langle \Phi_i \rangle_{i \in N})|_{\{p\}}$, then for all $\varphi \in X$, $F_1(\langle \Phi_i \rangle_{i \in N})|_{\{\varphi\}} \Leftrightarrow F_2(\langle \Phi_i \rangle_{i \in N})|_{\{\varphi\}}$.

Proof. Take any $q \in X \cap \Phi_0$. Define the profile $\langle \Phi'_i \rangle_{i \in N}$ in terms of $\langle \Phi_i \rangle_{i \in N}$ as follows: for all $i \in N$, $\Phi'_i = \Phi_i$ except at p where $\Phi'_i|_{\{p\}} \Leftrightarrow \Phi_i|_{\{q\}}$. Then

$$\begin{aligned}
 F_1(\langle \Phi_i \rangle_{i \in N})|_{\{q\}} &\Leftrightarrow F_1(\langle \Phi_i|_{\{q\}} \rangle_{i \in N}) && \text{by LI} \\
 &\Leftrightarrow F_1(\langle \Phi'_i|_{\{p\}} \rangle_{i \in N}) && \text{by LS} \\
 &\Leftrightarrow F_1(\langle \Phi'_i \rangle_{i \in N})|_{\{p\}} && \text{by LI} \\
 &\Leftrightarrow F_2(\langle \Phi'_i \rangle_{i \in N})|_{\{p\}} && \text{by hypothesis} \\
 &\Leftrightarrow F_2(\langle \Phi'_i|_{\{p\}} \rangle_{i \in N}) && \text{by LI} \\
 &\Leftrightarrow F_2(\langle \Phi_i|_{\{q\}} \rangle_{i \in N}) && \text{by LS} \\
 &\Leftrightarrow F_2(\langle \Phi_i \rangle_{i \in N})|_{\{q\}} && \text{by LI.}
 \end{aligned}$$

It is clear that $F_1(\langle \Phi_i \rangle_{i \in N}) \cap \Phi_0 = F_2(\langle \Phi_i \rangle_{i \in N}) \cap \Phi_0$.

Hence, $F_1(\langle \Phi_i \rangle_{i \in N}) = F_2(\langle \Phi_i \rangle_{i \in N})$. ■

This proposition says that if two LS rules make the same judgment (acceptance, rejection, and abstention) on a propositional variable in the agenda, then they make the same judgment on all the formulas in the agenda. In the other words, an LS rule is determined by its responses to all the possible judgments on a fixed propositional variable in the agenda.

4.3. Computational Complexity

In this section, we study the *computational complexity of winner determination* for the literal-based lexicographic rule (Endriss et al. 2012), i.e., how hard is it for this rule to compute the result of a given profile of individual judgment sets. Formally, the decision problem of winner determination is formulated as follows: *for a given formula $\varphi \in X$ and a given profile $\langle \Phi_i \rangle_{i \in N}$, determine whether or not $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$.*

By the notation P, we denote the class of decision problems each of which is solved in a deterministic turing machine using a polynomial amount of computation time. Now, let us present the complexity bound.

Proposition 5. The winner determination for the literal-based lexicographic rule is in P.

Proof. On the one hand, we use Algorithm 1 to compute the set of collective judgments for literals. It is not difficult to check that Algorithm 1 can be implemented in time $O(m \times n^2)$, where m is the cardinality of X_0 and n is the number of agents. On the other hand, deciding whether a given proposition is accepted by the group amounts to a model-checking problem, which was proved to be in ALOGTIME by Buss (1987). Note that ALOGTIME is a complexity class that is subsumed by P. Combining it with the previous result, we then obtain the desired bound. ■

Algorithm 1 Determining Collective Judgments for Literals**Input:** A set of literals X_0 ; a profile $\langle \Phi_i \rangle_{i \in N}$;**Output:** A set of literals Y .

```

1:  $Y \leftarrow \emptyset$ ;
2: for all  $\alpha \in X_0$  do
3:   if for all  $i \in N$ ,  $\neg\alpha \notin \Phi_i$  or there is  $j \in N$  with  $i < j$  such that  $\alpha \in \Phi_j$ , and there
   is  $k \in N$  such that  $\alpha \in \Phi_k$  then
4:      $Y \leftarrow Y \cup \{\alpha\}$ ;
5:   end if
6: end for
7: return  $Y$ .

```

5. MULTIPLE-LEVEL COLLECTIVE DECISION MAKING

In this section, we first show that the literal-based lexicographic rule is not oligarchic in the standard sense and then demonstrate that with abstentions, oligarchic aggregation is not necessary to be a single-level determination, but can be a multiple-level collective decision making.

5.1. Oligarchy

One may be surprised to find that as an *unfair* aggregation rule, F is non-dictatorial by Proposition 3. In fact, this indicates that ND is a very weak condition imposed on judgment aggregation rules when abstention is allowable. Indeed Gärdenfors (2006) as well as Dietrich and List (2008a) showed that by giving up completeness, oligarchies instead of dictatorships are obtained. We next investigate in the new setting whether this proposed rule is oligarchic or not. We first recall the definition of an oligarchic rule in Dietrich and List (2008a) and Gärdenfors (2006) as follows:

Definition 4. An aggregation rule G satisfying UD is a weak oligarchy if there is a non-empty smallest subset $M \subseteq N$ such that for every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$,

$$\bigcap_{i \in M} \Phi_i \subseteq G(\langle \Phi_i \rangle_{i \in N}).$$

And an oligarchic rule G is strict if for every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$,

$$\bigcap_{i \in M} \Phi_i = G(\langle \Phi_i \rangle_{i \in N}).$$

In this case, we call G to be weakly (strictly) oligarchic w.r.t. M .

Intuitively, an aggregation rule satisfying universal domain is said to be weakly oligarchic if there is a non-empty smallest set M such that for any profile of individual judgment set, the group judgment set contains all the propositions if they are in every member's judgment set of M . Furthermore, an aggregation rule is strictly oligarchic if

for any profile of individual judgment set, the group judgment set is exactly the set of propositions that are in every member's judgment set of M . Special cases of weak oligarchic aggregation rules are unanimous ($M = N$) and dictatorial ($M = \{i\}$) rules. More specifically, they are both weakly and strictly oligarchic. However, the literal-based lexicographic rule F is neither weakly oligarchic nor strictly oligarchic. Here is a simple counterexample.

Example 3. Let $N = \{1, 2\}$ with $\leq = \emptyset$, $X = \{p, q, p \rightarrow q, \neg p, \neg q, \neg(p \rightarrow q)\}$ and $X_0 = \{p, q, \neg p, \neg q\}$. Individual judgment set for each agent is given as follows: $\Phi_1 = \{p, q, p \rightarrow q\}$ and $\Phi_2 = \{\neg p, \neg q, p \rightarrow q\}$. Then $\Phi_1 \cap \Phi_2 = \{p \rightarrow q\}$, but according to the literal-based aggregation rule F , we have that $p \# F(\langle \Phi_1, \Phi_2 \rangle)$, $q \# F(\langle \Phi_1, \Phi_2 \rangle)$ and $p \rightarrow q \# F(\langle \Phi_1, \Phi_2 \rangle)$. Thus, $\Phi_1 \cap \Phi_2 \not\subseteq F(\langle \Phi_1, \Phi_2 \rangle)$.

It may be a bit surprising to find that the literal-based aggregation rule is not oligarchic. In fact, this does not violate the results in Dietrich and List (2008a) and Gärdenfors (2006), because their conditions imposed on the aggregation rules are more strengthened than ours. Specifically, their unanimity and systemacity conditions hold for all formulas, while we restrict them to literals. The proposed rule is literal-based, and the compound formulas are dependent on the collective judgments on the literals. That is, if a formula is a logical consequence of the collective judgments on literals, then it belongs to the set of collective judgments; otherwise, it is undetermined. Therefore, instead of the whole agenda, we need to consider the oligarchy notion with respect to the set of literals in the agenda instead of the whole agenda. This idea leads to a weak concept of oligarchy as follows:

Definition 5. An aggregation rule G satisfying UD is weakly oligarchic w.r.t. X_0 if there is a non-empty smallest subset $M \subseteq N$ such that for every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$,

$$\{\varphi \in X \mid \bigcap_{i \in M} \Phi_i \cap X_0 \models \varphi\} \subseteq G(\langle \Phi_i \rangle_{i \in N}).$$

And an oligarchic rule G is strict w.r.t. X_0 if for every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$,

$$\{\varphi \in X \mid \bigcap_{i \in M} \Phi_i \cap X_0 \models \varphi\} = G(\langle \Phi_i \rangle_{i \in N}).$$

This definition says that an aggregation rule satisfying universal domain is said to be weakly oligarchic w.r.t. X_0 if there is a non-empty smallest set M such that for any profile of individual judgment set, the group judgment set contains all the consequences of literals that are in every member's judgment set of M . Similarly, an aggregation rule is strictly oligarchic w.r.t. X_0 if for any profile of individual judgment set, the group judgment set is exactly the set of consequences of the literals that are in every member's judgment set of M .

We have the following proposition saying the literal-based lexicographic rule F is weakly oligarchic w.r.t. X_0 , but not strictly oligarchic w.r.t. X_0 .

Proposition 6. The literal-based lexicographic rule F is weakly oligarchic w.r.t. X_0 , but not strictly oligarchic w.r.t. X_0 .

Proof. Suppose F satisfies universal domain, it suffices to find a non-empty smallest set $M \subseteq N$, such that for every profile of individual judgment sets $\langle \Phi_i \rangle_{i \in N}$, every $\varphi \in X$, if $\bigcap_{i \in M} \Phi_i \cap X_0 \models \varphi$, then $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$. Let $O = \text{Max}_{\geq}(N) = \{i \in N : \nexists j \in N. i < j\}$. Because $(N, <)$ is well prioritized and $|N| \geq 2$; thus, such O must exist and should be non-empty. Suppose for every profile of individual judgment sets $\langle \Phi_i \rangle_{i \in N}$, every $i \in N$ and for all $\alpha \in X_0$, if $\alpha \in \bigcap_{i \in O} \Phi_i$, then according to Definition 1, $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$. Then $\bigcap_{i \in O} \Phi_i \cap X_0 \subseteq F(\langle \Phi_i \rangle_{i \in N})_0$. Because $\bigcap_{i \in O} \Phi_i \cap X_0 \models \varphi$, thus, $F(\langle \Phi_i \rangle_{i \in N})_0 \models \varphi$; thus, $\varphi \in F(\langle \Phi_i \rangle_{i \in N})$ by Definition 3.

We next show O is the smallest one with $\{\varphi \in X \mid \bigcap_{i \in O} \Phi_i \cap X_0 \models \varphi\} \subseteq F(\langle \Phi_i \rangle_{i \in N})$. Suppose not, then there is some $A \subseteq N$ such that $A \subset O$ and $\{\varphi \in X \mid \bigcap_{i \in A} \Phi_i \cap X_0 \models \varphi\} \subseteq F(\langle \Phi_i \rangle_{i \in N})$, then there is some $a \in N$ such that $a \in O$ but $a \notin A$. Take some $\beta \in X_0$ and define $\beta \in \Phi_i$ for every $i \in N \setminus \{a\}$ and $\neg\beta \in \Phi_a$, then by Definition 1, $\beta \notin F(\langle \Phi_i \rangle_{i \in N})$, but $\beta \in \{\varphi \in X \mid \bigcap_{i \in A} \Phi_i \cap X_0 \models \varphi\}$, contradicting with the assumption. Thus, O is just the required M .

It is easy to construct a profile such that $\bigcap_{i \in M} \Phi_i \cap X_0 \not\subseteq F(\langle \Phi_i \rangle_{i \in N})$ by LU. Take $\alpha \in X_0$ and define $\alpha \# \Phi_a$ for some $a \in M$ and $\alpha \in \Phi_x$ for every $x \in N \setminus \{a\}$. By LU with abstentions, $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$, but $\alpha \notin \bigcap_{i \in M} \Phi_i \cap X_0$. Thus, F is not strictly oligarchic w.r.t. X_0 . ■

On the one hand, the literal-based lexicographic aggregation rule F is only oligarchic with respect to the subset of literals in the agenda instead of the whole agenda because of its literal-based characteristics. On the other hand, we refine the oligarchy property, and this method might be generalized to distinguish other aggregation rules.

5.2. Multiple-Level Collective Decision Making

In this subsection, we show that with abstentions oligarchic aggregation is not necessary to be a single-level determination but can be a multiple-level collective decision making.

Before presenting the result, we need the following notion. We say that a set D of agents is *decisive* for a judgment aggregation rule G if every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$, $\bigcap_{j \in D} \Phi_j \subseteq G(\langle \Phi_i \rangle_{i \in N})$. Note that a weakly (strictly) oligarchic set is a decisive set, but not the converse, because a decisive set may not be the smallest. We further restrict a decisive set to a specific literal $\alpha \in X_0$ as follows:

Definition 6. A set of agents D is decisive on $\alpha \in X_0$ for a judgment aggregation rule G iff for every profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(G)$, if $\alpha \in \Phi_j$ for every $j \in D$, then $\alpha \in G(\langle \Phi_i \rangle_{i \in N})$.

It follows that the set $\text{Max}_{\geq}(N) = \{i \in N : \nexists j \in N. i < j\}$ is decisive for G , yet the decisive set for G is not unique, such as N . Next, we partition the hierarchical group N by classifying individuals with the same level into one subgroup from top to bottom according to $<$.

Definition 7. Let $<$ be a hierarchy on N . Then induced by $<$, N can be partitioned into subgroups M_1, \dots, M_n , where $\emptyset \neq M_i \subseteq N$ for every $i \in N$, $\bigcup_{i=1}^n M_i = N$ and M_i is inductively defined as follows:

- $M_1 = \{i \in N : \nexists j \in N. i < j\}$
- $M_{k+1} = \{i \in N \setminus (\bigcup_{i=1}^k M_i) : \nexists j \in N \setminus (\bigcup_{i=1}^k M_i). i < j\}$

It is clear that for every $l, k \in \{1, \dots, n\}$, if $l \neq k$, then $M_l \cap M_k = \emptyset$. Thus, we can say that this is a partition of N in terms of the hierarchical levels. If the group N can be

partitioned into n subgroups by earlier definition, we say the height of the hierarchical group N is n , denoted by $h(N) = n$. For every $i \in N$, if $i \in M_k$, we say the rank of individual i is k . For example, if $a \in M_1$, then a is at the level 1, i.e., at the top of the hierarchy.

The following result displays that a literal is not rejected by every of the superiors is sufficient and necessary to make the subgroup composed of the immediate inferiors a decisive set on this literal.

Proposition 7. Given a hierarchy among N with $h(N) = n$, let M_1, \dots, M_n be the subgroups of each level, for every $k \in \{0, \dots, n - 1\}$ and $\alpha \in X_0$, M_{k+1} is decisive on α for the literal-based lexicographic rule F if and only if $\neg\alpha \notin \Phi_i$ for every $i \in \bigcup_{h=0}^k M_h$.

Proof. It suffices to prove this result by induction on k .

- If $k = 0$, it is trivial as $M_1 = \{i \in N : \nexists j \in N. i < j\}$ is decisive on α for F and $\bigcup M_0 = \emptyset$.
- If $k = l + 1$, where $0 \leq l \leq n - 1$. Suppose $\neg\alpha \notin \Phi_i$ for every $i \in \bigcup_{h=0}^{l+1} M_h$, and given arbitrary profile $\langle \Phi_i \rangle_{i \in N} \in \text{Dom}(F)$, $\alpha \in \Phi_j$ for every $j \in M_{l+2}$, want $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$. Further assume $\neg\alpha \in \Phi_m$ for every $m \in N$, then $m \notin \bigcup_{h=0}^{l+2} M_h$, according to the definition of subgroups, there must be some superior j for m in M_{l+2} with $\alpha \in \Phi_j$, thus by the definition of F , $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$. Conversely, suppose M_{l+2} is decisive on α for F , and there is some $a \in \bigcup_{h=0}^{l+1} M_h$ such that $\neg\alpha \in \Phi_a$. And define a profile $\langle \Phi_i \rangle_{i \in N}$, where $\neg\alpha \in \Phi_a$, $\alpha \in \Phi_j$ for every $j \in M_{l+2}$ and $\alpha \notin \Phi_l$ for every $l \in N \setminus M_{l+2} \setminus \{a\}$, then $\alpha \notin F(\langle \Phi_i \rangle_{i \in N})$. However from assumption that M_{l+2} is decisive on α for F , and $\alpha \in \Phi_j$ for every $j \in M_{l+2}$, we have $\alpha \in F(\langle \Phi_i \rangle_{i \in N})$: contradiction.

Thus, the result holds. ■

Alternatively, this indicates that non-oligarchs can have the power to make collective decisions on some atomic proposition if and only if the proposition is not rejected by the oligarchs, which partially displays that with abstentions; oligarchic aggregation is no longer a single-level determination but can also be a multiple-level collective decision making.

6. DISCUSSION

In this article, we have provided a hierarchical approach to deal with priorities over individuals and proposed a feasible rule for judgment aggregation with abstentions under voters' hierarchy. Meanwhile, we have identified an impossibility result in this new setting and explored a set of plausible conditions for aggregation rules with respect to abstentions. In addition, we have shown that the proposed rule satisfies the plausible conditions and investigated its computational complexity as well as the oligarchic property.

6.1. Related Work

A growing body of literature on judgment aggregation has emerged in recent years. For an overview of the related research, see List and Puppe (2009), List and Polak (2010), List (2012), Grossi and Pigozzi (2014), and Endriss (2016). In the following, we will review the literature that is most related to our work.

To the best of our knowledge, Dietrich and List (2008b) is the first work to study expert rights or liberal rights in the context of judgment aggregation. It identified a liberal paradox for judgment aggregation and explored special conditions to avoid the paradox. We obtain an impossibility result: ignoring abstentions in decision process may lead to inconsistent collective judgments. It is worth noting that there is no direct relation between the two impossibility results, as we use a hierarchical approach to treat priorities over individuals without involving the concept of individual rights in Dietrich and List (2008b). In our setting, the individual with the highest hierarchy (if unique) has the individual rights over the whole agenda, yet there is no guarantee that two individuals have the rights to decide (at least) one proposition in the agenda. This is called *the minimal rights* in Dietrich and List (2008b). Violating of the minimal rights is one of the reasons why the proposed rule avoids the liberal impossibility.

The idea to employ the lexicographic method for dealing with judgment aggregation under individuals' hierarchy is inspired by Andréka et al. (2002) where a generalization of the lexicographic rule for combining ordering relations had been theoretically studied. In particular, Andréka et al. (2002) applied the lexicographic rule to preference aggregation in social choice and showed that the lexicographic rule was the only way of combining preference relations that satisfies a set of plausible conditions. Different from their motivation, we aim to propose a specific procedure for judgment aggregation under voters' hierarchy so as to investigate how the individual judgments affects the group judgment in a hierarchical environment. It turns out that the lexicographic procedure works well for this purpose. On the one hand, our work may be regarded as the first attempt to apply the lexicographic rule for judgment aggregation, which expands the application domain of this rule; on the other hand, our results enrich the theoretical study of the lexicographic rule. We also investigate the computational complexity of this rule, which is not involved in Andréka et al. (2002).

To circumvent the impossibility result in Theorem 1, we require the lexicographic judgment aggregation rule to be literal-based. To some extent, the proposed rule may be regarded as a special case of the premise-based rule (Dietrich and Mongin 2010; List 2012). Dietrich and Mongin (2010) provided a premise-based approach to judgment aggregation. In their framework for judgment aggregation, some formulas of the agenda satisfying special conditions were singled out as premises. Their definition of premises is more general. As a special case, we take the set of literals as the premise due to the principle of compositionality. Different from our motivation and approach, Dietrich and Mongin (2010) mainly studied necessary and sufficient conditions under which the combination of the premise-based rule and the conclusion-based rule leads to dictatorship oligarchy, respectively.

6.2. Applications

We do believe that this framework is general enough for handling applications out of judgment aggregation. Let us mention three typical problems where it is relevant.

- In reason-based theory of rational choice, an agent's preferences are determined by his or her motivating reasons, together with a *weighing relation* between different combinations of reasons (Dietrich and List 2013b). For instance, when buying a property, we may express our preference in the way of specifying which locations we like the house to be, how many bedrooms we want it to have, and which price range we can afford. Unfortunately, not all of the reasons can be satisfied. Then we need to sort the reasons. Accordingly, the reason-based social choice is to aggregate individuals' reasons into collective reasons so as to determine the collective choice. Most often, reasons are represented in the form of propositions, and the weighing relation can be treated as a hierarchy

over propositions. This allows us to reduce the problem of combining reasons with weighing relations to the problem of judgment aggregation under individuals' hierarchy.

- In database systems, prioritized queries can be used to describe suboptimal results (Fagin 2002). For instance, one may start with a query such as find a second-hand house less than 5 years near subway, if subway is unavailable then near bus stop. A compound prioritized query, which consists of at least two prioritized queries, can be dealt with by collective rules. Moreover, the data aggregation algorithm combines information that may be provided by different agents so as to produce a top- k list. The combination of prioritized results issued from multiple agents can then be reduced to the aggregation of individual judgments. The proposed rule is likely to play a role in this process.
- In belief merging, the aim is to combine several pieces of information coming from different sources in a unique one (Konieczny and Pérez 2011). One of the key issues for merging multiple knowledge bases is to deal with logical inconsistency. Even though each knowledge base is consistent, putting them together may give rise to inconsistent one. Thus, the aggregation procedure in belief merging faces problems close to those addressed in judgment aggregation. It has been widely accepted that priorities play an important role for resolving inconsistency when multiple knowledge bases are merged. Our framework is flexible to represent the priority information and to deal with inconsistency. In particular, we believe that the proposed rule can be used as a merging operator for prioritized knowledge bases (Qi et al. 2006; Qi 2007).

6.3. Future Work

As an attempt to provide a hierarchical approach for judgment aggregation, directions of future research are manifold.

First, as a special kind of lexicographic rule, it is interesting to investigate a representation result for our proposed rule. The lexicographic rule has been extensively studied in preference aggregation, and Andréka et al. (2002) proved that lexicographic rule is the only way of combining preference relations satisfying some natural conditions that are very close to Arrow's conditions. In particular, the property proved in Proposition 4 paves the way for the canonicity result of the lexicographic rule in Andréka et al. (2002). We envisage it would play a similar role for the characterization result of the literal-based lexicographic rule.

Second, under the provision of abstentions, we have investigated a set of commonly desirable conditions. It is natural to investigate some possibility results with respect to these conditions. Comparing with the impossibility results, we obtain a possibility result: This seems positive news to the result in Mongin (2008). Unfortunately, it is not the case because we do not assume completeness at both individual and collective levels.

Third, with abstentions, the dictatorship in judgment aggregation can also vary in degrees (Rossi et al. 2005). It is highly interesting to investigate the possibility scope between rationality and dictatorship under a set of plausible conditions. Some work has been performed in this direction (Dietrich and List 2013a).

Last but not least, since the universal and perfect aggregation rule does not exist, a plausible way is to boil down aggregation problems to three-sided matching questions among specific degenerate rules, specific groups, and specific agendas, which is a promising direction (Dietrich and List 2008b).

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