Frame problem in dynamic logic

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ABSTRACT. *This paper provides a formal analysis on the solutions of the frame problem by using dynamic logic. We encode Pednault's syntax-based solution, Baker's state-minimization policy, and Gelfond & Lifchitz's Action Language $\mathcal{A}$ in the propositional dynamic logic (PDL). The formal relationships among these solutions are given. The results of the paper show that dynamic logic, as one of the formalisms for reasoning about dynamic domains, can be used as a formal tool for comparing, analyzing and unifying logics of action.*

KEYWORDS: *dynamic logic, frame problem, reasoning about action.*

1. Introduction

The theory of action and change has been a focus of AI for over thirty years. Most fundamental problems in this area, such as the frame problem, ramification problem, and qualification problem, have been widely investigated with varying degrees of success [MCC 69, GIN 88, BAK 91, REI 91, LIN 91, LIN 94, SAN 94, LIN 95, MCC 95, MCC 97, SHA 97, THI 97, GEL 98, CAS 99]. As a result, many formal systems have established in last few decades. Among them are Situation Calculus [MCC 69, REI 91], STRIPS [FIK 71], Dynamic Logic [HAR 79, KOZ 90], Event Calculus [KOW 86], Action Languages [GEL 98], Fluent Calculus [THI 97] and a large number of other monotonic or nonmonotonic logics such as in [GIN 88]. The time has come to analyze, compare and systematize these formalisms and solutions in order to obtain a more complete and unified (if possible) theory of action. This paper focuses on the solutions to the frame problem. We compare and analyze the main solutions to the frame problem in the literature by encoding them in the propositional dynamic logic (PDL).

In spite of a great diversity of languages and inference mechanisms, the existing solutions to the frame problem can be grouped into two categories: *monotonic* and *nonmonotonic*.

Monotonic approaches seek automatic procedures to generate frame axioms from effect axioms based on certain meta-hypothesis such as the *Explanation Closure Assumption* or the *Causal Completeness Assumption* [SCH 90, PED 89, REI 91, GIA 95, THI 97, CAS 99].

Nonmonotonic approaches attempt to introduce new inference mechanisms based on the meta-hypothesis such as the *common sense law of inertia* or *minimization*, to capture defaults about the effects of actions implied by frame axioms [BAK 91, SAN 94, LIN 95, SHA 97, MCC 97, GEL 98, GIN 88].

It could rise to much controversy in how to map the existing solutions to each of the categories. However, it is easy to see that most of the existing solutions can be traced back to the three typical solutions: *Pednault's syntax-based solution* [PED 89], *Baker's circumscription* [BAK 91], and *the Action Language* $\mathcal{A}$ [GEL 93]. For instance, Reiter's solution is an extension of Pednault's approach [REI 91]. Most of circumscription-based solutions were developed on Baker's approach [LIN 95, SHA 97, GIN 88]. To simplify the analysis and comparison of these solutions, we will consider only the above mentioned three solutions to the frame problem. Some pioneer work along this direction has been done in the literature. [KAR 93] provided a sound and completed translation from Action Language A [GEL 93] to Pednault's, Reiter's and Baker's formalisms, which shows that these three solutions to the frame problem are equivalent provided the domain under consideration can be expressed in Action Language $\mathcal{A}$. As it has been pointed out in [KAR 93], Action Language $\mathcal{A}$ is rather limited in its expressive power. In fact it is the least expressive action description language among the existing action formalisms. Kartha's result shows the similarity of the above mentioned solutions but the approach is unable to differentiate them from each other. To make a systematic comparison of the action formalisms, we need a formal system with high expressive power in action description and well-established inference mechanism and associated semantic model in order to bridge all the action formalisms. As one of the formalisms for reasoning about action and change, dynamic logic(DL) offers many prominent features that makes it possible to meet the purpose.

Firstly, dynamic logic language is highly expressive which allows to represent any complicated actions[1], non-deterministic effects and action qualifications (See Appendix for the examples of intertranslation between different languages). It can also be extended to represent concurrent actions [GIO 98], non-execution of actions [GIA 95], indirect effects of actions [GIO 00, ZHA 01] and causal relations [ZHA 01]. Secondly, dynamic logic features a sound and complete axiomatic deductive system and a well-developed Kripkean semantics. The proof theory and model theory of dy-

1. Dynamic logic can express any programs or compound actions by using program connectives $;, \cup, ?, and*$ (See [HAR 79] for more details).

namic logic have reached a high degree of sophistication through the development of theoretical computer science. Some features, such as decidability and the finite model property, and techniques such as bisimulation and filtration, are well understood. In contrast to other action logics, dynamic logic does not offer built-in solutions to the frame problem. It has been shown that some existing approaches, no matter whether they are monotonic or nonmonotonic, can be transplanted to DL or PDL [GIA 95, PRE 96, CAS 99]. This paper does not aim to bring up a new solution of the frame problem with dynamic logic [2]. Instead, we take advantage of the high expressive power of dynamic logic language to encode each of the above mentioned solutions in dynamic logic language and compare these solutions through different ways of minimization of PDL models.

The paper is organized as follows. The next section will give a brief introduction to the action logic based on the extended PDL [ZHA 01]. Section 3 will present some properties of PDL models under certain syntactical restrictions on action descriptions and queries. Section 4, 5 and 6 will translate Pednault's, Gelfond and Lifschitz's and Baker's solutions to the minimization of PDL model, respectively. This translation provides a bridge to link all the approaches together.

2. Reasoning about actions in PDL

2.1. *Preliminaries*

To make the exploration simple, we will only consider the propositional case of dynamic logic. A propositional dynamic logic(PDL) language consists of a set Flu of fluent symbols (propositional variables) and a set Act_P of primitive action symbols. We will use f, f_1, f_2, etc., to denote fluents, and use a, a_1 ,a_2, etc., for primitive actions. The formulas ($A \in Fma$) and actions ($\alpha \in Act$) can be defined as usual [HAR 79, KOZ 90]. A formula which does not include modal operators is referred to as a *propositional formula* ($\varphi \in Fma_P$). In dynamic logic, a causal relation between an action α and its effect A is expressed by a modal formula: $[\alpha]A$, read as "*α always causes A*". For instance, $[Shoot]\neg alive$ represents "*shoot at a turkey kills the turkey*". The formula $\langle\alpha\rangle A$ reads as "*α is executable and possibly causes A to be true*", where $\langle\alpha\rangle$ is the dual operator of $[\alpha]$. In particular, $\langle\alpha\rangle\top$ represents "*α is executable*", where $\top$ represents the logical constant **true**. $\prec a \succ A$ denotes "$\langle a\rangle\top \rightarrow \langle a\rangle A$", meaning "*if a is executable, then a may cause A.* " The semantics and deductive system of PDL can be found in any standard introductory text e.g [HAR 79, KOZ 90, GOL 87].

2.2. *Action description*

PDL provides a formal language to describe behaviors and internal relations of a dynamic system. Those sentences which describe the generic effects of actions,

2. See [ZHA 02b] for such a solution.

domain constraints and causal ramifications are generally called *action description* or *domain description*, which can be represented by any (finite or infinite) set of PDL formulas.

EXAMPLE 1. — Consider the Yale Shooting Problem [HAN 87] described by the following action description:

$$\Sigma = \left\{ \begin{array}{l} \neg loaded \rightarrow [Load]loaded \\ loaded \rightarrow [Shoot]\neg alive \\ loaded \rightarrow [Shoot]\neg loaded \\ \langle Load \rangle \top \\ \langle Wait \rangle \top \\ \langle Shoot \rangle \top \end{array} \right\}$$

The first three statements describe the effects of the actions: $Load$ and $Shoot$ on fluent $loaded$ and $alive$ (effect axioms). The last three represent the executability of actions (qualification axioms). Note that the action description does not include frame axioms. Therefore reasoning with the action description requires a solution of the frame problem. □

2.3. ***Normal action description***

An action description can be any set of formulas. However, in most cases an action description can be converted to a simple normal form.

An action description Σ is *normal* if each formula in Σ is of the form:

– $\varphi \rightarrow [a]l$ (*deterministic action law*)

– $\varphi \rightarrow \prec a \succ l$ (*non-deterministic action law*)

– $\varphi \rightarrow \langle a \rangle \top$ (*qualification law*)

where $\varphi \in Fma_P$, $a \in Act_P$ and l is a fluent literal.[3]

It is easy to see that the action description in Example 1 is normal.

We remark that not every action description can be converted to a normal form. For instant, a formula like $[a][a]l$ is unable to be expressed by normal form. However, normal form is still quite expressive. It is easy to prove that any action description written in the form of pre-condition axioms and successor state axioms in the *propositional* situation calculus language (that is, there are no sort *object* and function symbols in the language [REI 91]) can be translated into normal form (see appendix). Action descriptions written in Action Language $\mathcal{A}$ or in STRIPS and the determinism of actions (i.e., for any initial state there exists one and only one next state) can also be expressed by normal form [ZHA 02a].

3. In [ZHA 02a], the normal form of action descriptions is defined in a more general version to express indirect effects of actions based on the extended PDL [ZHA 01].

2.4. *Reasoning with action description*

An action description describes generic effects of action and domain constraints. For instance, the sentence “$loaded \rightarrow [Shoot]\neg alive$” says that whenever $loaded$ is true, $Shoot$ causes $\neg alive$. In the language of situation calculus, this is written as $\forall s(loaded(s) \rightarrow \neg alive(do(Shoot, s)))$. Indeed, we need to view the action description of a dynamic domain as a set of extra axioms (domain axioms in the situation calculus [REI 91]) rather than an ordinary set of formulas in reasoning about the domain [ZHA 01].

DEFINITION 2 ([ZHA 01]). — *Let Σ be an action description. A formula A is Σ-*provable*, written $\vdash^{\Sigma} A$, if it belongs to the smallest set of formulas which contains all theorems of PDL, all elements of Σ, and is closed under* modus ponens *and* necessitation (modal generalization) *[GOL 87].*

Consider the action description Σ in Example 1. We prove that $\vdash^{\Sigma} \neg loaded \rightarrow [Load; Shoot]\neg alive$ as follows:

(1) $\vdash^{\Sigma} loaded \rightarrow [Shoot]\neg alive$ (AD)

(2) $\vdash^{\Sigma} [Load](loaded \rightarrow [Shoot]\neg alive)$ (1 and N)

(3) $\vdash^{\Sigma} [Load]loaded \rightarrow [Load][Shoot]\neg alive$ (2 and K)

(4) $\vdash^{\Sigma} [Load]loaded \rightarrow [Load; Shoot]\neg alive$ (3 and $Comp$)

(5) $\vdash^{\Sigma} \neg loaded \rightarrow [Load]loaded$ (AD)

(6) $\vdash^{\Sigma} \neg loaded \rightarrow [Load; Shoot]\neg alive$ (4 and 5)

where AD indicates action description in Σ; $Comp$ is an axiom of PDL; N and K represent *the Necessitation* and *K axiom*, respectively. MP denotes *modus ponens*.

3. Properties of PDL models

We now present some special properties of PDL models which are not included in the standard discourse of dynamic logic but are useful for the purpose of the paper.

3.1. PDL *models*

A model for a PDL language is a structure of the form $M = (W, \{R_a : a \in Act_P\}, V)$, where W is a non-empty set of possible worlds, for each primitive action a, R_a is a binary relation on W, and V is a function mapping from Flu to W. Note that we only consider the accessibility relations of primitive actions. The accessibility relations for compound actions can be defined by using the standard model conditions [GOL 87]. The satisfiability relation is defined as usual. A model M *satisfying* a formula A in a world w is denoted $M \models_w A$. A is *valid* in M, denoted by $M \models A$, if $M \models_w A$ for all $w \in W$. Let Σ be an action description. A model M is a Σ-*model* if $M \models A$ for any $A \in \Sigma$. Intuitively, a model is a Σ-model if Σ is true in every world of the model; $Mod(\Sigma)$ denotes the set of all Σ-models. In [ZHA 01], it is shown that *for any action description* Σ, $\vdash^{\Sigma} A$ *iff* A *is valid in all* Σ*-models.* We now investigate models which are relevant to the models of action languages and situation calculus.

3.2. *Consistency of action description*

As we have seen, an action description acts as an axiomatic specification of a dynamic system highlighting the importance of consistency. In [ZHA 02a] three different levels of system consistency: *consistency of formulas, consistency of action descriptions* and *consistency of formulas with action descriptions* were discussed. Each of these conveys different information about the dynamic system under consideration. For the purpose of this paper, we only consider the consistency of action descriptions.

An action description Σ is *consistent*[4] if $\not\vdash^{\Sigma} \bot$, where $\bot$ represents the logical constant "**false**". The following result follows directly from the soundness and completeness of Σ-provability (Theorem 2 in [ZHA 01]):

PROPOSITION 3 ([ZHA 01]). — *An action description* Σ *is consistent iff there exists a* Σ*-model.*

Now let's consider the consistency of normal action description. Let Σ be an action description in normal form. For any fluent f and any primitive action a, if we merge the action laws about a and f in each form together, there are at most five causal laws about a and f in Σ:

$$\varphi_0 \to \langle a\rangle\top$$

$$\varphi_{1,1} \to [a]f, \varphi_{1,2} \to [a]\neg f$$

$$\varphi_{2,1} \to \prec a \succ \neg f, \varphi_{2,2} \to \prec a \succ f$$

It is easy to see that if φ_0, $\varphi_{1,1}$and $\varphi_{1,2}$ are true simultaneously, then the action description will contain a contradiction. Similarly for φ_0, $\varphi_{1,j}$ and $\varphi_{2,j}$ ($j = 1$ or $j = 2$). This leads to the following definition.

DEFINITION 4. — *A normal action description* Σ *is* safe *if* Σ *satisfies the following assumption: for any fluent* f *and primitive action* a*,*

$\vdash \neg\varphi_0 \vee \neg\varphi_{1,1} \vee\neg\varphi_{1,2}$ *and* $\vdash \neg\varphi_0 \vee \neg\varphi_{1,j} \vee\neg\varphi_{2,j}$ *(*$j = 1, 2$*)*

where φ_0, $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$, *and* $\varphi_{2,2}$ *are the outcomes of the above merging process.*

It is easy to verify that the action description in Example 1 is safe.

4. In [ZHA 02a], it is called *uniformly consistent*, distinguishing from the consistency of ordinary formulas.

The following proposition shows that if an action description is represented in normal form, the consistency checking can be much easy.

PROPOSITION 5. — *Let Σ be a normal action description. If Σ is safe, then it is consistent.*

PROOF. — Given an action description Σ, let Σ^* be a variant of Σ which is generated by the following procedure:

Step 1: Set $\Sigma = \Sigma^*$ and for each primitive action a,

1) if there is no qualification law in the form $\varphi \rightarrow \langle a \rangle \top$ in Σ, then let $\bot \rightarrow \langle a \rangle \top \in \Sigma^*$;

2) for each fluent literal l, if there is no deterministic action law in the form $\varphi \rightarrow [a]l$ in Σ, add $\bot \rightarrow [a]l$ to Σ^*.

3) for each fluent literal l, if there is no non-deterministic action law in the form $\varphi \rightarrow \prec a \succ l$ in Σ, add $\bot \rightarrow \prec a \succ l$ to Σ^*.

Step 2: for each primitive action a and fluent literal l, suppose that all the action laws in Σ^* about a and l are:

$$\varphi_0 \rightarrow \langle a \rangle \top$$

$$\varphi_{1,1} \rightarrow [a]l, \ \ \varphi_{1,2} \rightarrow [a]\neg l$$

$$\varphi_{2,1} \rightarrow \prec a \succ \neg l, \ \ \varphi_{2,2} \rightarrow \prec a \succ l$$

Then, we replace $\varphi_{2,1} \rightarrow \prec a \succ \bar{l}$ by $(\neg \varphi_0 \vee \neg \varphi_{1,1}) \rightarrow \prec a \succ \bar{l}$, and $\varphi_{2,2} \rightarrow \prec a \succ l$ by $(\neg \varphi_0 \vee \neg \varphi_{1,2}) \rightarrow \prec a \succ l$, where $\bar{l}$ denotes the dual literal of l.

We term Σ^* the *completion* of Σ. It is easy to verify that

1) if Σ is normal and safe, then Σ^* is;

2) if Σ^* is consistent, then Σ is.

Therefore, to prove the consistency of Σ, we can assume that $\Sigma = \Sigma^*$ without loss of generality. For such an action description Σ, we construct a standard structure $M = (W, \{R_\alpha : \alpha \in Act_P\}, V)$ as follows:

1) W ={$w : w$ is an interpretation of Flu}.

2) For each primitive action $a \in Act_P$, $(w, w') \in R_a$ iff

- there exists $\varphi \rightarrow \langle a \rangle \top \in \Sigma$ such that $w \models_{PL} \varphi$[5].
- for every $\varphi \rightarrow [a]l \in \Sigma$, if $w \models_{PL} \varphi$, then $w' \models_{PL} l$; and
- there exists $\varphi \rightarrow \prec a \succ l \in \Sigma$ such that if $w \models_{PL} \varphi$ then $w' \models_{PL} l$.

3) For any compound action $\alpha \in Act$, R_α is given inductively by the standard model condition on α[GOL 87].

4) For any primitive proposition p, $V(p) = \{w : w \models_{PL} p\}$.

5. Here $w \models_{PL} \varphi$ means φ is a logical consequence of w under classical propositional logic.

Now we prove that Σ is valid in M.

1. Suppose that $\varphi \to [a]l \in \Sigma$. For any $w \in W$, if $M \models_w \varphi$, then $w \models_{PL} \varphi$. Thus for any $w' \in W$ with wR_aw', by the construction of M, $w' \models_{PL} l$, so $M \models_{w'} l$. Therefore $M \models \varphi \to [a]l$.

2. Suppose that $\varphi_0 \to \prec a \succ l_0 \in \Sigma$. For any $w \in W$, if $M \models_w \varphi_0$, then $w \models_{PL} \varphi_0$. Let $\varphi_1 \to \langle a \rangle \top$ be the qualification law for a (by the construction of action description completion, the qualification law always exists). For any $w \in W$, if $M \not\models_w \varphi_1$, then $w \not\models_{PL} \varphi_1$. According to the construction of M, there is no $w' \in W$ such that $(w, w') \in R_a$, thus $M \models_w \prec a \succ l_0$, that is, $M \models_w \varphi_0 \to \prec a \succ l_0$; otherwise $M \models_w \varphi_1$, then $w \models_{PL} \varphi_1$. Let $H = \{l : \exists\, \varphi \to [a]l \in \Sigma(w \models_{PL} \varphi)\} \cup \{l_0\}$. We now prove that H is consistent. To this end, suppose that there is a conflict f and $\neg f$ in H.

Case 1: $f = l_0$ or $\neg f = l_0$, say the former, there must exist a law $\varphi_2 \to [a]\neg f \in \Sigma$. According to the assumption of safety, $\vdash \neg\varphi_0 \vee \neg\varphi_1 \vee \neg\varphi_2$. We know that $w \models_{PL} \varphi_0 \wedge \varphi_1$, so $w \not\models_{PL} \varphi_2$. Thus $\neg f \notin H$, a contradiction.

Case 2: $f \neq l_0$ and $\neg f \neq l_0$, then there must be action laws $\varphi_2 \to [a]f$ and $\varphi_3 \to [a]\neg f$ in Σ. According to the assumption of safety again, $\vdash \neg\varphi_1 \vee \neg\varphi_2 \vee \neg\varphi_3$. We know that $w \models_{PL} \varphi_1$, thus, $w \models \neg\varphi_2 \vee \neg\varphi_3$. This contradicts both f and $\neg f$ in H.

Finally, we extend H into w' such that for any fluent $f \in Flu$, $f \in w'$ iff $\neg f \notin w'$. That means w' is an interpretation of Flu. It is easy to see that $w' \in W$ and $(w, w') \in R_a$. Therefore, $M \models_w \varphi_0 \to \prec a \succ l_0$.

3. For any $\varphi_0 \to \langle a \rangle \top \in \Sigma$, suppose that $M \models_w \varphi_0$, that is, $w \models_{PL} \varphi_0$. Similar to the non-deterministic law, there is $w' \in W$ such that $(w, w') \in R_a$, therefore $M \models_w \langle a \rangle \top$, or $M \models_w \varphi_0 \to \langle a \rangle \top$. We conclude that M is a Σ-model. ■

3.3. ***Saturated models***

From this section, we will introduce several restrictions on PDL models in order to match the expressive power of PDL models and other action models.

DEFINITION 6. — *A model $M = (W, \mathcal{R}, V)$ is* saturated *if for each interpretation I of Flu, there exists $w \in W$ such that $M \models_w I$. For any action description Σ, We use $Mod_S(\Sigma)$ to denote the set of all saturated Σ-models.*

PROPOSITION 7. — *If Σ is normal and safe, then for any formula A,*

$$\vdash^{\Sigma} A \text{ iff } M \models A \text{ for any } M \in Mod_S(\Sigma). \quad (1)$$

PROOF. —

"$\Rightarrow$" immediately follows Proposition 5 and the soundness of Σ-provability [ZHA 01].

"$\Leftarrow$" According to the completeness of Σ -provability [ZHA 01], we only have to prove that for any interpretation I of Flu, there exists a maximal Σ-consistent set which includes I. This can be easy done by using the approach we used in the proof of Proposition 5. ■

We will show in section 6 that the saturation of PDL models corresponds to the Existence of Situation Axioms (ESA) [BAK 91]. Note that Proposition 7 depends on the definition of the normal form of action description. If we allow an action description to describe domain constraints or indirect effects, this proposition will cease to hold.

EXAMPLE 8. — Let $Flu = \{f_1, f_2\}$ and $Act_P = \varnothing$. Let $\Sigma = \{f_1 \to \neg f_2\}$. Obviously $Mod_S(\Sigma) = \varnothing$. Therefore for any $M \in Mod_S(\Sigma)$, $M \models \bot$. However, $\not\vdash^{\Sigma} \bot$. □

3.4. *Natural models*

DEFINITION 9. — *A model $M = (W, \mathcal{R}, V)$ is* natural *if*

1) W is the set of all interpretations of Flu,

2) for any $f \in Flu$, $w \in V(f)$ iff $f \in w$.

We denote the set of all natural Σ-models by $Mod_N(\Sigma)$.

It is easy to see that any natural model is saturated but it is not true conversely.

We call a formula to be a *simple query* if the formula is represented in the form $\varphi \to [a_1; \cdots; a_n]l$, where φ is a propositional formula, l a literal.

PROPOSITION 10. — *Let Σ be a normal and safe action description. For any simple query $A = \varphi \to [a_1; \cdots; a_n]l$,*

$$\vdash^{\Sigma} A \text{ iff } M \models A \text{ for any } M \in Mod_N(\Sigma). \quad (2)$$

PROOF. — It is easy to see that we only need to prove that whenever $\not\vdash^{\Sigma} \varphi \to [a_1; \cdots; a_n]l$, there exists a natural Σ-model to falsify it. According to Proposition 7, there is a natural Σ-model $M = (W, \mathcal{R}, V)$ which can falsify it, that is, there is a world w_0 such that $M \models_{w_0} \varphi \wedge \neg[a_1; \cdots; a_n]l$. For each $w \in W$, let $h(w)$ denote the interpretation of Flu which satisfies $M \models_w h(w)$(Note that $h(w)$ is unique). Let $M' = (W', \mathcal{R}', V')$ where

1) $W' = \{h(w) : w \in W\}$,

2) $(I_1, I_2) \in R'_a$ iff there exists w_1 and w_2 such that $I_1 = h(w_1)$, $I_2 = h(w_2)$ and $(w_1, w_2) \in R_a$,

3) $I \in V'(f)$ iff $f \in I$.

It is not too hard to prove by induction that for any formula ψ, $M \models_w \psi$ iff $M' \models_{h(w)} \psi$. Now we prove M' is a Σ-model.

Suppose that $\varphi \to \langle a\rangle\top \in \Sigma$. For any $h(w) \in W'$, if $M' \models_{h(w)} \varphi$, then $M \models_w \varphi$. Since M is a Σ-model, $M \models_w \langle a\rangle\top$. Hence there exists $w' \in W$ such that $(w, w') \in R_a$. Thus $(h(w), h(w')) \in R'_a$, which means $M' \models_{h(w')} \langle a\rangle\top$.

Suppose that $\varphi \to [a]l \in \Sigma$. For any $h(w) \in W'$, if $M' \models_{h(w)} \varphi$, then $M \models_w \varphi$. For any $I \in W'$ such that $(h(w), I) \in R'_a$, let $h(w) = h(w_1)$, $I = h(w_2)$ and $(w_1, w_2) \in R_a$. It follows from $M \models_w \varphi$ that $M \models_{w_1} \varphi$. Since M is a Σ -model, we yield $M \models_{w_2} l$, so $M' \models_I l$. Therefore $M' \models_{h(w)} [a]l$.

Suppose that $\varphi \to\prec a \succ l \in \Sigma$. For any $h(w) \in W'$, if $M' \models_{h(w)} \varphi$, then $M \models_w \varphi$. If there is $(h(w), I) \in R'_a$, there exist w_1, w_2 such that $h(w) = h(w_1)$, $I = h(w_2)$ and $(w_1, w_2) \in R_a$. It follow from $M \models_w \varphi$ that $M \models_{w_1} \varphi$. Since M is a Σ-model, there exists w'_2 such that $(w_1, w'_2) \in R_a$ and $M \models_{w'_2} l$. Thus $M' \models_{h(w'_2)} l$. Therefore $M' \models_{h(w)} \varphi \to\prec a \succ l$.

Finally, we prove that M' falsifies $\varphi \to [a_1; \cdots; a_n]l$. Since $M \models_{w_0} \varphi$, we have $M' \models_{h(w_0)} \varphi$. From $M \models_{w_0} \neg[a_1; \cdots; a_n]l$ we know that there exists $w_1, \cdots, w_n$ such that $(w_0, w_1) \in R_{a_1}, \cdots, (w_{n-1}, w_n) \in R_{a_n}$ and $M \models_{w_n} \neg l$. It follows that $(h(w_0), h(w_1)) \in R'_{a_1}, \cdots, (h(w_{n-1}), h(w_n)) \in R'_{a_n}$ and $M' \models_{h(w_n)} \neg l$. Therefore $M' \models_{h(w_0)} \varphi \wedge \neg[a_1; \cdots; a_n]l$. ■

Notice that Proposition 10 is true only for *simple queries*. The following example shows that if we allow A to be any formula, the expression 2 will not be necessarily true.

EXAMPLE 11. — Let $\Sigma = \varnothing$, $Flu = \{f\}$ and $Act_P = \{a\}$. Let $A = f \vee \prec a \succ \neg f \vee \prec a \succ\prec a \succ \neg f \vee \prec a \succ\prec a \succ\prec a \succ f$. It is not too hard to verify that A is valid in all the natural models(by checking all possible binary relations on $W = \{\{f\}, \{\neg f\}\}$). However, the following non-natural model can falsify A:

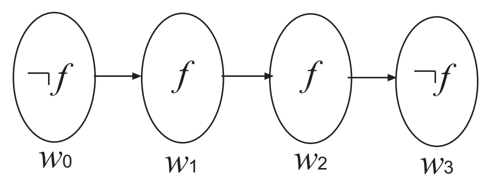

Figure 1. *A non-natural model that falsifies the statement* $f \vee \prec a \succ \neg f \vee \prec a \succ\prec a \succ \neg f \vee \prec a \succ\prec a \succ\prec a \succ f$.

□

3.5. *Functional models*

DEFINITION 12. — *A model* $M = (W, \mathcal{R}, V)$ *is* functional *if for any* $a \in Act_P$, R_a *is a function on* W. *We write* $Mod_F(\Sigma)$ *for the set of all functional* Σ*-models and* $Mod_{NF}(\Sigma)$ *for* $Mod_N(\Sigma) \cap Mod_F(\Sigma)$.

The syntactical condition with respect to functional models is so-called *determinism*, which means that each state can have and only have one next state after performing an action.

DEFINITION 13. — *Let* $\Xi = \{\langle a\rangle f \to [a]f : a \in Act_P$ *and* $f \in Flu\} \cup \{\langle a\rangle\top : a \in Act_P\}$. *An action description is* deterministic *if* $\Sigma \vdash \Xi$.[6]

Note that $\langle a\rangle f \to [a]f$ can be converted to normal form as follows:

$$f' \to [a]f, \neg f' \to [a]\neg f$$

where f' is a new fluent symbol (in most cases, we can put the descriptions of determinism and effects of actions together without introducing new fluent symbols).

PROPOSITION 14. — *Let* Σ *be a normal and safe action description. If* Σ *is deterministic, then*

$$\vdash^{\Sigma} A \text{ iff } M \models A \text{ for any } M \in Mod_{NF}(\Sigma) \tag{3}$$

PROOF. — It is easy to see that we only need to prove that whenever $\not\vdash^{\Sigma} A$, there exists a natural functional Σ-model to falsify it. According to Proposition 7, there is a saturated Σ-model $M = (W, \mathcal{R}, V)$ which can falsify it. By the proof of Proposition 10, we can reduce this model to a natural Σ-model $M' = (W', \mathcal{R}', V')$. We only need to check if M' is functional and can falsify A. According to the proof of Proposition 10, there exists a function h such that for any $w \in W$, $h(w) \in W'$ and $M \models_w h(w)$.

Since M' is a Σ-model, it is a Ξ-model. Therefore M' is functional. To show M' can falsify A, we prove that $M \models_w B$ iff $M' \models_{h(w)} B$ by induction on B's structure.

1. B has the form $\langle a\rangle C$. If $M \models_w \langle a\rangle C$, there exists w' such that $(w, w') \in R_a$ and $M \models_{w'} C$. It follows that $(h(w), h(w')) \in R'_a$. By inductive hypothesis, $M' \models_{h(w')} C$. Thus $M' \models_{h(w)} \langle a\rangle C$. Conversely, suppose that $M' \models_{h(w)} \langle a\rangle C$. There exists I such that $(h(w), I) \in R'_a$ and $M' \models_I C$. Since M is a Ξ-model, there exists w' such that $(w, w') \in R_a$. If $I \neq h(w')$, M' will not be a Ξ-model. So $I = h(w')$. This means $M \models_w \langle a\rangle C$.

2. B has the form $[a]C$. Assume that $M \models_w [a]C$. For any $(h(w), I) \in R'_a$, there exists w' such that $(w, w') \in R_a$ and $I = h(w')$. Then $M \models_{w'} C$. By inductive hypothesis, $M' \models_{h(w')} C$. Thus $M' \models_{h(w)} [a]C$. The other direction is similar.

It is easy to verify the other cases. ■

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6. We assume here that a deterministic action is always executable. The condition can be relaxed at a price of more complex formalization.

Note the difference between Proposition 10 and 14. We can relax the restriction of simple query at the price of allowing only deterministic action descriptions.

Put the above propositions together, we can easily see the effects of syntactical restrictions on PDL models:

Let Σ be a normal, safe and deterministic action description. Let A be a simple query. Then $\vdash^{\Sigma} A$ iff $M \models A$ for any $M \in Mod_S(\Sigma) \cap M \in Mod_N(\Sigma) \cap M \in Mod_F(\Sigma)$.

3.6. *Minimizing PDL models*

Before we encode any solution of the frame problem in dynamic model, let's see how to minimize changes in dynamic logic models.

Let $M = (W, \mathcal{R}, V)$ be a PDL model. For any $w \in W$, let $||w|| = \{f \in Flu : M \models_w f\} \cup \{\neg f : f \in Flu \;\&\; M \not\models_w f\}$. We denote $Chg(M) = \{(a, f, w) : \exists w'(wR_a w' \;\&\; f \in (||w|| \backslash ||w'||) \cup (||w'|| \backslash ||w||)\}$. In words, $(a, f, w) \in Chg(M)$ iff there exists an accessible world w' to w on action a such that the truth value of f is different at w and w'.

DEFINITION 15. — *For any M_1, $M_2 \in Mod(\Sigma)$, $M_1 \sqsubset M_2$ iff*

1) $W_1 = W_2$,

2) $V_1(f) = V_2(f)$,

3) $Chg(M_1) \subset Chg(M_2)$.

We denote the set of $\sqsubset$-minimal models in $Mod(\Sigma)$ by $\min(Mod(\Sigma))$. Intuitively, $M_1 \sqsubset M_2$ means M_1 has fewer state changes than M_2.

4. Pednault's solution to the frame problem

We first encode Pednault's syntax-based solution [PED 89] to the frame problem in PDL. Before doing this, let's recall the meaning of the frame problem.

To formalize the effects of actions in a dynamic system, it is necessary to provide all the effect axioms of actions (which specify what is affected by actions). Often this is easy because most actions affect only a few of the relevant fluents. In contrast, listing all the frame axioms (which specify what is not affected by actions) is tedious. Moreover, they are much more numerous than effect axioms. For instance, in Example 1, only effect axioms were listed. There are nine frame axioms, such as $alive \to [Load]alive$, $loaded \to [Wait]loaded$ etc., that were not listed. Without these axioms, the action description is incomplete. We cannot even establish the intuitive assertion $\vdash^{\Sigma} alive \to [Load]alive$. The frame problem is how to invent an inference mechanism for reasoning about effect of action with incomplete action descriptions.

Pednault [PED 89] introduced an approach to the frame problem with which frame axioms can be automatically generated from effect axioms and qualification axioms. Consider a normal action description Σ without non-deterministic action laws. Suppose that the positive and negative effect axioms and qualification axioms about action a and fluent f in an action description Σ are:

$$\varphi_0 \to \langle a \rangle \top,\, \varphi_1 \to [a]f,\, \varphi_2 \to [a]\neg f.$$

According to the *Causal Completeness Assumption* [REI 91], we have the following frame axioms:

$FA^+_{a,f} : (\neg\varphi_0 \vee \neg\varphi_2) \wedge f \to [a]f$

$FA^-_{a,f} : (\neg\varphi_0 \vee \neg\varphi_1) \wedge \neg f \to [a]\neg f$

All frame axioms generated by this procedure are referred to as *the frame axioms with respect to* Σ. For instance, $\neg loaded \wedge alive \to [Shoot]alive$ is a frame axiom about $Shoot$ and $alive$ with respect to the action description in Example 1. Suppose that Δ is the set of all the generated frame axioms with respect to Σ. Then we are able to prove that $\{\neg loaded\} \vdash^{\Sigma\cup\Delta} [Load; Wait; Shoot]\neg alive$, where Δ is the set of all frame axioms.

In general, given a set Σ of effect axioms and qualification axioms, we generate all the frame axioms with the above procedure. Let Δ be all the generated frame axioms. Then $\Sigma \cup \Delta$ will be the complete action description with respect to Σ. Therefore, to answer a query A, we only have to make the inference $\vdash^{\Sigma\cup\Delta} A$.

LEMMA 16. — *Let Σ be a normal and safe action description without non-deterministic action laws. Then*

$$\min(Mod_S(\Sigma)) = Mod_S(\Sigma \cup \Delta),$$

where Δ is the set of all frame axioms with respect to Σ.

PROOF. — "$\subseteq$". Assume that $M \in \min(Mod_S(\Sigma))$. We prove that $M \models \Delta$.

For any primitive action a and fluent f, suppose that the action laws about a and f in Σ are:

$\varphi_0 \to \langle a \rangle \top$

$\varphi_1 \to [a]f$

$\varphi_2 \to [a]\neg f$

then the frame axioms about a and f are

$FA1_{a,f}$: $(\neg\varphi_0 \vee \neg\varphi_2) \wedge f \to [a]f$

$FA2_{a,f}$: $(\neg\varphi_0 \vee \neg\varphi_1) \wedge \neg f \to [a]\neg f$

Suppose that $M \not\models FA1_{a,f}$. That means there exist w_0 and w_1 such that $w_0 R_a w_1$, $M \models_{w_0} (\neg\varphi_0 \vee \neg\varphi_2) \wedge f$ and $M \models_{w_1} \neg f$. It follows that $(a, f, w_0) \in Chg(M)$. Let $M' \in Mod(\Sigma)$ which is as same as M except

$$R'_a = \begin{cases} (R_a \backslash \{(w_0, w') : w' \in W\}) \cup \{(w_0, w_2)\}, & \text{if } M \models_{w_0} \varphi_0; \\ R_a \backslash \{(w_0, w') : w' \in W\}, & \text{otherwise.} \end{cases}$$

where w_2 satisfies $M \models_{w_2} (||w_1||_M \backslash \{\neg f\}) \cup \{f\}$. The existence of w_2 is due to that M is a saturated PDL model and so can satisfy any interpretation of Flu (Note that $(||w_1||_M \backslash \{\neg f\}) \cup \{f\}$ is an interpretation of Flu).

Now we prove that M' is still a Σ-model. In fact, we only need to check if the action laws about action a are satisfied at w_0 in M'. It is easy to see that for any propositional formula ψ, $M \models_w \psi$ iff $M' \models_w \psi$.

If $M \models_{w_0} \neg\varphi_0$, then $\{w' : (w_0, w') \in R'_a\} = \varnothing$. In this case, all the action laws about a are trivially true at w_0 in M'. So we can assume that $M \models_{w_0} \varphi_0$. With this in mind,

1. for the action law $\varphi_0 \to \langle a \rangle \top$, $M' \models_{w_0} \langle a \rangle \top$ holds because $(w_0, w_2) \in R'_a$.

2. for any action law $\varphi \to [a]l \in \Sigma$, if $M' \models_{w_0} \varphi$, then $M \models_{w_0} \varphi_0 \wedge \varphi$. It follows that $l \neq \neg f$. Since M is a Σ-model, $M \models_{w_0} l$, or, $l \in ||w_0||_M$. Therefore $l \in ||w_2||_{M'}$. That means $M' \models_{w_0} \varphi \to [a]l$.

Therefore M' is a Σ-model.

It is obvious that $Chg(M') \subseteq Chg(M)$. Since $(a, f, w_0) \notin Chg(M')$, we yield $M' \sqsubset M$, which contradicts that M is $\sqsubset$-minimal. Thus M is a model of $FA1_{a,f}$. Similar to $FA2_{a,f}$.

"$\supseteq$". Let $M \in Mod_S(\Sigma \cup \Delta)$. We again suppose that the action laws about a and f in Σ are:

$\varphi_0 \to \langle a \rangle \top$

$\varphi_1 \to [a]f$

$\varphi_2 \to [a]\neg f$

The associated frame axioms are:

$FA1_{a,f}$: $(\neg\varphi_0 \vee \neg\varphi_2) \wedge f \to [a]f$

$FA2_{a,f}$: $(\neg\varphi_0 \vee \neg\varphi_1) \wedge \neg f \to [a]\neg f$

Now suppose that there exists a natural Σ-model $M' \in \min(Mod_S(\Sigma))$ such that $M' \sqsubset M$. In other words, there is $(a, f, ||w||) \in Chg(M) \backslash Chg(M')$ such that for any w' which satisfies wR_aw' and $f \in (||w|| \backslash ||w'||) \cup (||w'|| \backslash ||w||)$. Without loss of generality, we assume that $M \models_w f$. Since M is a model of $FA1_{a,f}$, $M \models_w \varphi_0 \wedge \varphi_2$, hence $M' \models_w \varphi_0 \wedge \varphi_2$. It follows that $M' \models_{w'} \neg f$. Therefore $(a, f, ||w||) \in Chg(M')$, a contradiction. We conclude that $M \in \min(Mod_S(\Sigma))$. ■

The following theorem establishes the semantic condition for Pednault's solution. It also gives the relationship between syntax-based and minimization-based approaches.

THEOREM 17. — *Let Σ be a normal action description without non-deterministic action laws. If Σ is safe, then $\vdash^{\Sigma\cup\Delta} A$ iff $M \models A$ for any $M \in \min(Mod_S(\Sigma))$, where Δ is the set of frame axioms with respect to Σ.*

PROOF. — Straightforward from Lemma 16 and Proposition 7. ∎

5. Encoding the action language $\mathcal{A}$ in PDL

The action languages [GEL 93] offer a simple and elegant solution to the frame problem. In this section, we show that the action language $\mathcal{A}$ can be also embedded into PDL. Our approach can also be extended the action languages $\mathcal{B}$ and $\mathcal{C}$ if we base on the extended propositional logic ($EPDL$) [ZHA 01].

An action description Σ in the language $\mathcal{A}$ is a set of expressions of the form [GEL 93]:

a **causes** l **if** φ,

where a is a primitive action, l is a fluent literal, and φ is a conjunction of literals. The state of a dynamic domain is expressed by a set of *axioms* of the form:

now l

A query in action language $\mathcal{A}$ is an expression of the form:

necessarily φ **after** $a_1, \cdots, a_n$,

where φ is a propositional formula and $a_1, \cdots, a_n$ are primitive actions.

A structure $T = (W, \{R_a \subseteq W \times W : a \in Act_P\}, V)$ is a *transition system* of an action description Σ if

1. W is the set of all interpretations of Flu,

2. V is a function from Flu to 2^W such that $f \in V(w)$ iff $f \in w$.

3. $(w, w') \in R_a$ iff $E(a, w) \subseteq w' \subseteq E(a, w) \cup w$, where $E(a, w)$ is the set of the head l of all expression "a **causes** l **if** φ" in Σ such that w satisfies φ.

Let Γ be a set of expressions in the form: **now** l. A query "**necessarily** φ **after** $a_1, \cdots, a_n$" is a consequence of Γ in T if, for any chain $(w_0, w_1) \in R_{a_1}, \cdots, (w_{n-1}, w_n) \in R_{a_n}$, whenever w_0 satisfies l for each **now** $l \in \Gamma$, w_n satisfies φ.

A query "**necessarily** φ **after** $a_1, \cdots, a_n$" is a consequence of Γ with respect to an action description Σ if it is a consequence of Γ in any transition system of Σ.

According to the translation between $\mathcal{A}$ and PDL shown in the Appendix, we can easily transform an action description and a state description between two languages.

Since such a translation is one-to-one, we will only use PDL language to describe action descriptions, initial states and queries. They are easily recognized with context. It is easy to see that an action description in language $\mathcal{A}$ is always normal and safe. There is an important difference between the semantics of action language and PDL. In $\mathcal{A}$, there is no explicit expression for qualification of actions. An underlying assumption, called *Qualification Completeness*, in the semantics is that *an action is always executable unless the action description implies that it is not.* In PDL, there is no such assumption. Thus qualification of actions must be explicitly specified.

Let Σ be a finite action description in $\mathcal{A}$. Suppose the action laws about an action a and fluent f are $\varphi_1 \to [a]f$ and $\varphi_2 \to [a]\neg f$. This implies that a is not executable when $\varphi_1 \wedge \varphi_2$. Collecting all the conditions of non-executability: $\varphi_1^1 \wedge \varphi_2^1, \cdots, \varphi_1^n \wedge \varphi_2^n$, we know that a is not executable if $(\varphi_1^1 \wedge \varphi_2^1) \vee \cdots \vee (\varphi_1^n \wedge \varphi_2^n)$. By *Qualification Completeness*, we assume that $(\neg(\varphi_1^1 \wedge \varphi_2^1) \wedge \cdots \wedge \neg(\varphi_1^n \wedge \varphi_2^n)) \to \langle a \rangle \top$; such a condition is an *induced qualification law* . Let Λ be the set of all such laws from Σ. Then we have

THEOREM 18. — *Let Σ be a finite action description and Γ a finite set of axioms, both in $\mathcal{A}$. A query "***necessarily** φ **after** $a_1, \cdots, a_n$*" is a consequence of Γ with respect to Σ iff $\vdash^{\Sigma \cup \Lambda \cup \Delta} (\bigwedge \Gamma) \to [a_1; \cdots; a_n]l$, where Δ is the set of the frame axioms with respect to $\Sigma \cup \Lambda$.*

PROOF. — According to Proposition 10, it is easy to see that we only have to prove that $M \in Mod_S(\Sigma \cup \Lambda \cup \Delta)$ iff M is a transition system of Σ.

Suppose that $M = (W, \mathcal{R}, V) \in Mod_S(\Sigma \cup \Lambda \cup \Delta)$. For each $(w, w') \in R_a$, since M is a Σ-model, $E(a, w) \subseteq w'$. Assume that $l \in w' \backslash (E(a, w) \cup w)$. Let the action laws about a and l (and its dual literal) in $\Sigma \cup \Lambda$ be:

$\varphi_0 \to \langle a \rangle \top$

$\varphi_1 \to [a]l$

$\varphi_2 \to [a]\bar{l}$

where $\bar{l}$ is the dual literal of l. Thus the frame axiom about a and $\bar{l}$:

$(\neg\varphi_0 \vee \neg\varphi_1) \wedge \bar{l} \to [a]\bar{l}$

Since $l \notin E(a, w)$, w does not satisfies φ_1. On the other hand, $l \notin w$ implies $\bar{l} \in w$. Hence w satisfies $(\neg\varphi_0 \vee \neg\varphi_1) \wedge \bar{l}$. Since M is a Δ-model, $\bar{l} \in w'$, a contradiction. Therefore $w' \subseteq E(a, w) \cup w$.

Conversely, assume that M is a transition system of Σ. It is easy to see that it is a natural Σ-model. Now we prove that it is a
$Lambda \cup \Delta$-model.

Let $\neg(\varphi_1^1 \wedge \varphi_2^1) \wedge \cdots \wedge \neg(\varphi_1^n \wedge \varphi_2^n) \to \langle a \rangle \top$ be the induced qualification axiom of a from Σ. If w satisfies $\neg(\varphi_1^1 \wedge \varphi_2^1) \wedge \cdots \wedge \neg(\varphi_1^n \wedge \varphi_2^n)$, then $E(a, w) = \{f : \varphi_1 \to [a]f \in \Sigma$ and $w \models_{PL} \{\varphi_1\} \cup \{\neg f : \varphi_2 \to [a]\neg f \in \Sigma$ and $w \models_{PL} \varphi_2\}$ is consistent, where $w \models_{PL} \varphi$ means w satisfies φ in classical propositional logic. Let

$w' = E(a, w) \cup \{f \in w : \neg f \notin E(a, w)\} \cup \{\neg f \in w : f \notin E(a, w)\}$. It is obvious that w' is an interpretation of Flu and $E(a, w) \subseteq w' \subseteq E(a, w) \cup w$. Therefore $(w, w') \in R_a$. That is $M \models_w \langle a \rangle \top$.

Suppose that the frame axioms about a and f are:

$(\neg\varphi_0 \vee \neg\varphi_2) \wedge f \rightarrow [a]f$,

$(\neg\varphi_0 \vee \neg\varphi_1) \wedge \neg f \rightarrow [a]\neg f$.

For any $(w, w') \in R_a$, if w satisfies $(\neg\varphi_0 \vee \neg\varphi_2) \wedge f$ but $f \notin w'$, then $\neg f \in E(a, w)$. It follows that w satisfies φ_2. Thus w falsifies φ_0. According to the construction of Λ, there exists a fluent f' such that $\varphi_1' \rightarrow [a]f'$, $\varphi_2' \rightarrow [a]\neg f' \in \Sigma$ and $w \models_{PL} \varphi_1' \wedge \varphi_2'$. It follows that f',$\neg f' \in E(a, w) \subseteq w'$, a contradiction. ■

Clearly, the expressive power of $\mathcal{A}$ is quite restricted. Action descriptions can only be normal and queries can only be simple in our terminology.

6. Encoding Baker's solution in PDL models

Finally we consider Baker's solution. First, we have to recall the basic assumption of the approach.

6.1. *Models of situation calculus*

A model of situation calculus (SC model in short) consists of the following three different kinds of domains[MCC 69, BAK 91]:

– a domain of situations: $|\mathcal{M}|_s$,

– a domain of actions: $|\mathcal{M}|_a$,

– a domain of fluents: $|\mathcal{M}|_f$.

as well as interpretations for special predicates:

1) the interpretations for the relations $Holds$ and Ab:
$Holds^{\mathcal{M}} \subseteq |\mathcal{M}|_f \times |\mathcal{M}|_s$,
$Ab^{\mathcal{M}} \subseteq |\mathcal{M}|_a \times |\mathcal{M}|_f \times |\mathcal{M}|_s$,

2) the interpretation for the $Result$ function:
$Result^{\mathcal{M}} \in (|\mathcal{M}|_a \times |\mathcal{M}|_s \rightarrow |\mathcal{M}|_s)$.

The following axioms were used in Baker's circumscriptive solution to the frame problem:

1) Unique names axioms:

- Unique Name Axioms for fluents (UNAF): *for any* $f_1, f_2 \in Flu$, $f_1 \neq f_2$.
- Unique Name Axioms for Actions(UNAA): *for any* $a_1, a_2 \in Act_P$, $a_1 \neq a_2$.

2) Commonsense Law of Inertia (CLI):

$$\neg Ab(a, f, s) \leftrightarrow (Holds(f, Result(a, s)) \leftrightarrow Holds(f, s))$$

3) Domain Closure Axioms:

- Domain Closure Axiom for Fluents (DCAF):

$$f = f_1 \vee f = f_2 \vee \cdots \vee f = f_n \vee \cdots$$

- Domain Closure Axiom for Actions (DCAA):

$$a = a_1 \vee a = a_2 \vee \cdots \vee a = a_n \vee \cdots$$

4) Existence of Situation Axioms (ESA):

$$\exists s(Holds(f_1, s) \wedge Holds(f_2, s) \wedge \cdots \wedge Holds(f_n, s) \wedge \cdots)$$

$$\exists s(Holds(f_1, s) \wedge \neg Holds(f_2, s) \wedge \cdots \wedge Holds(f_n, s) \wedge \cdots)$$

$$\cdots$$

$$\exists s(\neg Holds(f_1, s) \wedge \neg Holds(f_2, s) \wedge \cdots \wedge \neg Holds(f_n, s) \wedge \cdots)$$

Since the $Holds$ predicate cannot be nested, not every formula in PDL can be translated into the situation calculus language. We call an action description to be SC-expressible if it can be translated into the situation calculus language. From now on, we will only consider *SC-expressible* action description. Whenever we say Σ to be an action description, it could be either a domain description in Situation Calculus or its translation in PDL, depending on the context. Furthermore, we call an SC model $\mathcal{M}$ to be a Σ-*model* when $\mathcal{M}$ satisfies all the formulas in Σ.

6.2. *Relationship between SC models and PDL models*

First, we translate an SC model to a PDL model.

DEFINITION 19. — *Let $\mathcal{M}$ be an SC model. A PDL model $M = (W, \mathcal{R}, V)$ is the* corresponding SC model *of $\mathcal{M}$ if*

1) $W = |\mathcal{M}|_s$,
2) $(s_1^{\mathcal{M}}, s_2^{\mathcal{M}}) \in R_a$ iff $s_2^{\mathcal{M}} = Result^{\mathcal{M}}(a^{\mathcal{M}}, s_1^{\mathcal{M}})$,
3) $s^{\mathcal{M}} \in V(f)$ iff $Holds^{\mathcal{M}}(f^{\mathcal{M}}, s^{\mathcal{M}})$.

Next, we consider the transformation of PDL models to SC models.

DEFINITION 20. — *Let $M = (W, \mathcal{R}, V)$ be a functional PDL model. An SC model $\mathcal{M}$ is the* corresponding model *of M if*

1) $|\mathcal{M}|_f = Flu$, $|\mathcal{M}|_a = Act_P$, $|\mathcal{M}|_s = W$, *2)* $s' = Result^{\mathcal{M}}(a,s)$ *iff* $(s,s') \in R_a$*,*

3) $(f,s) \in Holds^{\mathcal{M}}$ *iff* $f \in V(s)$*,*

4) $(a,f,s) \in Ab^{\mathcal{M}}$ *iff* $(a,f,s) \in Chg(M)$*.*

We have finished the translation between an SC model and a PDL model. The following lemmas establish the relationship between two kinds of models. The proofs are straightforward from the above definitions.

LEMMA 21. — *Let* $M = (W, \mathcal{R}, V)$ *be the corresponding* PDL *model of an SC model* $\mathcal{M}$*. Then*

1) $M \models_{s^{\mathcal{M}}} \varphi$ *iff* $\mathcal{M} \models Holds(\varphi, s)$*.*

2) M *is functional.*

3) If $\mathcal{M}$ *satisfies the common sense law of inertia, then* $(a, f, s^{\mathcal{M}}) \in Chg(M)$ *iff* $(a^{\mathcal{M}}, f^{\mathcal{M}}, s^{\mathcal{M}}) \in Ab^{\mathcal{M}}$*.*

4) If Σ *is a normal action description, then* M *is a* Σ*-model iff* $\mathcal{M}$ *is a* Σ*-model.*

5) If $\mathcal{M}$ *satisfies the Existence of Situation Axioms, then* M *is saturated.*

LEMMA 22. — *Let* $\mathcal{M}$ *be the corresponding SC model of a functional* PDL *model* $M = (W, \mathcal{R}, V)$*. Then*

1. $\mathcal{M} \models Holds(\varphi, s)$ *iff* $M \models_{s^{\mathcal{M}}} \varphi$

2. $\mathcal{M}$ *satisfies the Commonsense Law of Inertia.*

3. $\mathcal{M}$ *satisfies the Domain Closure Axioms for Fluents and Actions.*

4. $\mathcal{M}$ *satisfies the Unique Names Axioms for Fluents and Actions.*

5. If Σ *is a normal action description, then* $\mathcal{M}$ *is a* Σ*-model iff* M *is a* Σ*-model.*

6. If M *is saturated, then* $\mathcal{M}$ *satisfies the Existence-of-Situation Axiom.*

LEMMA 23. — *Let* M *be a functional* PDL *model. If* $\mathcal{M}$ *is the corresponding SC model of* M*, then* M *is the corresponding PDL model of* $\mathcal{M}$*. Conversely, suppose that* $\mathcal{M}$ *is an SC model and* M *the corresponding PDL model of* $\mathcal{M}$*. If* $\mathcal{M}$ *satisfies:*

1) Domain Closure Axioms for Fluents and Actions,

2) Unique Names Axioms for Fluents and Actions,

then $\mathcal{M}$ *is the corresponding SC model of* M*.*

6.3. *Relationship between Baker's circumscription policy and* PDL*-model-based minimization*

The following theorem sets up the link between Baker's circumscription policy and PDL-model-based minimization.

THEOREM 24. — *Let* Σ *be a deterministic and SC-expressible action description.*

1) $M \in \min(Mod_{NF}(\Sigma))$ *iff its corresponding* SC *model is a model of* $CIRCUM(\Sigma \cup \Psi; Ab; Result)$*.*

2) $\mathcal{M}$ *is a model of* $CIRCUM(\Sigma \cup \Psi; Ab; Result)$ *if and only if its corresponding SC model belongs to* $\min(Mod_{NF}(\Sigma))$*.*

where Ψ *is the set of UNAF, UNAA, DCAF and DCAA.*

PROOF. — We only prove the first statement. The second one is similar.

"$\Leftarrow$". Let $M = (W, \mathcal{R}, V)$ be the corresponding PDL model of a SC model $\mathcal{M}$ which satisfies $CIRCUM(\Sigma \cup \Psi; Ab; Result)$. If M is not minimal, there exists a Σ-model M' such that $M' \sqsubset M$.

Let $\mathcal{M}'$ be the corresponding model of M'. By Lemma 22, $\mathcal{M}'$ is also a Σ-model. According to Definition 20, we know that

1) $|\mathcal{M}'|_s = |\mathcal{M}|_s$, $|\mathcal{M}'|_a = |\mathcal{M}|_a$, $|\mathcal{M}'|_f = |\mathcal{M}|_f$.

2) $Holds^{\mathcal{M}'} = Holds^{\mathcal{M}}$.

For any $w \in W$, $(a, f, w) \in Ab^{\mathcal{M}'}$, there exists w' such that $(w, w') \in R'_a$ and $f \in (||w||_{M'} \backslash ||w'||_{M'}) \cup (||w'||_{M'} \backslash ||w||_{M'})$.

Since $M' \sqsubset M$, we have $f \in (||w||_M \backslash ||w'||_M) \cup (||w'||_M \backslash ||w||_M)$. This means $(a, f, w) \in Ab^{\mathcal{M}}$. Therefore $Ab^{\mathcal{M}'} \subseteq Ab^{\mathcal{M}}$. On the other hand, $M' \sqsubset M$ means that there exist a, f, w_1 such that $(a, f, w_1) \in Chg(M) \backslash Chg(M')$, say $(w_1, w_2) \in R_a$ and $f \in (||w_1||_M \backslash ||w_2||_M) \cup (||w_2||_M \backslash ||w_1||_M)$.

Thus $(a, f, w_1) \in Ab^{\mathcal{M}}$. However, $(a, f, w_1) \notin Ab^{\mathcal{M}}$. Therefore $\mathcal{M}$ is not a model of $CIRCUM(\Sigma \cup \Psi; Ab; Result)$, a contradiction. We conclude $M \in \min(Mod_F(\Sigma))$.

"$\Rightarrow$". Let $M = (W, \mathcal{R}, V) \in \min(Mod_F(\Sigma))$ and $\mathcal{M}$ be the corresponding SC model of M. We prove by contraposition that $\mathcal{M}$ is a model of $CIRCUM(\Sigma \cup \Psi; Ab; Result)$. Assume that $\mathcal{M}'$ is a Σ-model of situation calculus with the properties:

1) $|\mathcal{M}'|_s = |\mathcal{M}|_s$, $|\mathcal{M}'|_a = |\mathcal{M}|_a$, $|\mathcal{M}'|_f = |\mathcal{M}|_f$,

2) $Holds^{\mathcal{M}'} = Holds^{\mathcal{M}}$ and $Ab^{\mathcal{M}'} \subset Ab^{\mathcal{M}}$.

Let $M' = (W', \mathcal{R}', V')$ is the corresponding PDL model of $\mathcal{M}'$. According to Lemma 23, M' is also a Σ-model.

Let's consider the relationship between M and M'.

First, it is easy to show that $W' = W$.

Second, $w \in V'(f)$ iff $(f, w) \in Holds^{\mathcal{M}'}$ iff $(f, w) \in Holds^{\mathcal{M}}$ iff $w \in V(f)$.

Finally, for any $(a, f, w) \in Chg(M')$, without lost of generality, assume that $(w_1, w_2) \in R'_a$ and $f \in (||w_1|| \backslash ||w_2||) \cup (||w_2|| \backslash ||w_1||)$. So $(a, f, w_1) \in Ab^{\mathcal{M}'}$, or $(a, f, w_1) \in Ab^{\mathcal{M}}$. It follows that $(a, f, w) \in Chg(M)$. Therefore $M' \sqsubset M$, which contradicts $M \in \min(Mod_F(\Sigma))$. ■

Note that the action description in the observation is not necessarily normal. However, if we impose syntactical restrictions on action description and queries, we can prove that all the solutions to the frame problem we considered thus far are equivalent. The following corollary of the above theorems coincides the result given by [KAR 93].

COROLLARY 25. — *Let Σ be a finite normal action description, Γ a finite set of literals. If Σ is deterministic and safe, then the following statements are equivalent:*

1. $\vdash^{\Sigma\cup\Delta} (\bigwedge\Gamma) \rightarrow [a_1;\cdots;a_n]l$, where Δ is the set of frame axioms with respect to Σ.

2. For any model $M \in \min(Mod_{NF}(\Sigma))$, $M \models (\bigwedge\Gamma) \rightarrow [a_1;\cdots;a_n]l$.

*3. "**necessarily** l **after** $a_1,\cdots,a_n$" is a consequence of Γ with respect to Σ.*

4. $CIRCUM(\Sigma\cup\Psi; Ab; Result) \models \forall s(Holds((\bigwedge\Gamma), s) \rightarrow Holds(l, Result(a_1,\cdots,a_n,s))).$

where Ψ is the set of UNAF, UNAA, DCAF, DCAA and ESA.

7. Conclusion and discussion

In this paper we presented a PDL-based approach to uniform the existing solutions of the frame problem. It has been shown that both monotonic approaches and nonmonotonic approaches can be encoded in PDL. Monotonic approaches can be simulated by automatically generating frame axioms with normalized action description in PDL. Nonmonotonic approaches can also be simulated by minimizing PDL models. These two different treatments to the frame problem can be nicely related by using the well-established PDL semantics. Based on the approach, we established a general relationship between three typical solutions to the frame problem: *Pednault's syntax-based solution*, *Baker's circumscription* and *Gelfond & Lifchitz's action language A*. This relationship not only shows the equivalence of these three solutions (similar to Kartha's result in [KAR 93]) but also discloses the differences of these approaches under different levels of syntactical or semantical restrictions.

With help of the formal results in the paper, we would like to make the following remarks:

Syntactical restrictions: The equivalence among the solutions to the frame problem relies heavily on the syntactical restrictions on action description and queries. For instance, if Σ is not normal, the validity of a formula A in all the natural Σ-models does not guarantee $\vdash^{\Sigma} A$. Thus the link between the Σ-provability in PDL and provability from transition systems of action language $\mathcal{A}$ will not exist. Additionally, the form of queries is also crucial to the equivalence. Fortunately, the link between minimizing PDL models and minimizing SC models does not depend on the normality of action description.

Extensibility of action formalisms: Each formalism of action has been or is intended to be extended to accommodate *non-deterministic and indirect effects* of actions, *compound actions* or *concurrent actions*. For such an extension, dynamic logic provides a foresightful paradigm. For instance, the extension of Situation Calculus in complex action representation can be easily traced to action compound in dynamic logic. Also, if we are going to extend $\mathcal{A}$ to answer a general query, the defined transition system should allow the existence of "non-natural" models (Proposition 10).

Epistemic minimization and physical minimization: We know that Baker's circumscriptive policy (varying $Result$) corresponds exactly to the minimization of PDL models. We may remember that we took a detour, varying $Holds$, before we reached the "right solution": state-minimization [SHA 97]. Such a detour does not seem necessary in PDL models or transition systems. There is a subtle difference between circumscriptive first-order models and minimizing PDL models. With circumscription we minimize abnormality whereas in PDL we minimize change of worlds. We refer to the former kind of minimization as to be *epistemic* and the latter as to be *physical*.

Lazy-formalization: Most solutions to the frame problem were based on the assumption that the formalization of a system should be strong enough to answer any query by direct inference. Since the number of frame axioms could combinatorially explode with unlimited numbers of fluents and actions, we can only either automatically generate all the frame axioms or invent new inference mechanisms to reason with frame axioms by default. [ZHA 02b] introduced a new approach that is able to release the assumption. The point is that if we can postpone formalize a dynamic system till a certain query arises, the number of frame axioms could be significantly reduced. It was shown that the frame axioms that actually needed for answering a query are those that syntactically relevant to the query. Therefore the number of frame axioms actually needed depends on the number of fluents and actions occurred in the query of interest rather than the whole system.

We have seen that dynamic logic can serve as a flexible and general platform for reasoning about action. Its high expressive syntax and well-established semantics provide a natural link for the existing formalisms for reasoning about action. The cross relationship between different solutions to the frame problem gives us a deeper understanding of the problem and the associated solutions. We believe that dynamic logic will play more important roles in reasoning about action and change.

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Appendix: Translations between languages

In this appendix We provide an intertranslation between dynamic logic, situation calculus and action languages. We use f to represent fluents, l for literals, a for actions and φ or ψ for any propositional formulas. We remark that this intertranslation is not formal. For instance, a fluent symbol stands for a proposition in PDL but is an individual in situation calculus. $Holds(\varphi, S_0)$ make sense only in the extended predicate of $Holds$. Additionally, all these translations depend on the semantics of the associated action logics.

1) Expressions for describing initial state:

Dynamic Logic	*Situation Calculus*	*Action Language $\mathcal{A}$*
f	$Holds(f, S_0)$	**now** f
$\neg f$	$\neg Holds(f, S_0)$	**now** $\neg f$
φ	$Holds(\varphi, S_0)$	

2) Expressions for describing queries:

Dynamic Logic	*Situation Calculus*	*Action Language $\mathcal{A}$*
$[a_1, \cdots, a_n]\varphi$	$Holds(\varphi, Result([a_1, \cdots, a_n], s))$	φ **after** $a_1, \cdots, a_n$

3) Expressions for describing causation between propositions:

Dynamic Logic	*Situation Calculus*	*Action Language $\mathcal{B}$*
$[\varphi]\psi$	$Holds(\varphi, s) \rightarrow Caused(\psi, true, s)$	ψ **if** φ

4) Expressions for describing domain axioms:

Dynamic Logic	*Situation Calculus*	*Action Language $\mathcal{C}$*
$\varphi \rightarrow [a]l$	$\forall s(Holds(\varphi, s) \rightarrow Holds(l, Result(a, s)))$	a **causes** l **if** φ
$\varphi \rightarrow \prec a \succ l$		a **may cause** l **if** φ
$\varphi \rightarrow \langle a \rangle \top$	$\forall s(Holds(\varphi, s) \rightarrow Poss(a, s))$	**executable** a **if** φ